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Nonperturbative superpotentials and compactification to three dimensions

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Abstract: We consider four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories with gauge group $\mathrm{U}(N)$ on $R^3 \times S^1$, in the presence of a classical superpotential. The low-energy quantum superpotential is obtained by simply replacing the adjoint scalar superfield in the classical superpotential by the Lax matrix of the integrable system that underlies the 4d field theory. We verify in a number of examples that the vacuum structure obtained in this way matches precisely that in 4d, although the degrees of freedom that appear are quite distinct. Several features of 4d field theories, such as the possibility of lifting vacua from $\mathrm{U}(N)$ to $\mathrm{U}(tN)$, become particularly simple in this framework. It turns out that supersymmetric vacua give rise to a reduction of the integrable system which contains information about the field theory but also about the Dijkgraaf-Vafa matrix model. The relation between the matrix model and the quantum superpotential on $R^3 \times S^1$ appears to involve a novel kind of mirror symmetry.

Keywords: Field Theories in Lower Dimensions, Nonperturbative Effects, Supersymmetric Effective Theories, Integrable Equations in Physics.

Contents

1. Introduction

The dynamics of supersymmetric gauge theories mimics in many ways that of ordinary QCD, allowing for instance for confinement and chiral symmetry breaking. Therefore, if we would have exact control over supersymmetric theories, we could imagine describing non-supersymmetric gauge theories as a perturbation away from a supersymmetric point, rather than as a perturbation around a free field theory. In view of the qualitative similarity of the low-energy dynamics, such a description could be much better behaved than ordinary perturbation theory. In addition, there are several indications that supersymmetry will be restored at sufficiently high energies, and this provides ample motivation for the study of supersymmetric gauge theories.

As a first step towards the complete understanding of supersymmetric gauge theories, one would like to understand their vacuum structure. A simple organizing principle to describe the vacuum structure, low-energy gauge couplings and correlation functions of chiral operators was given by Dijkgraaf and Vafa in [1, 2, 3]. This organizing principle involves a matrix model which is obtained from the classical superpotential by reducing it to its zero modes. The sum over the planar diagrams of the matrix model computes the quantum superpotential as a function of gluino condensate superfields S_i, one for each semisimple gauge group factor that is left unbroken by a choice of minimum of the classical superpotential. Though this result was originally derived using topological string theory, one can give a proof of it directly in perturbation theory [4], or alternatively using the Konishi anomaly [5].

The matrix model, which was originally found for a pure $\mathcal{N}=2$ theory deformed by a superpotential $\mathrm{Tr}\, W(\Phi)$, has been successfully generalized to a variety of other supersymmetric gauge theories. There are, however, still several conceptual questions that remain. One of these is whether and why the gluino condensate superfields play such a distinguished role. It is true, as has been elaborated in [5], that they effectively control the complete chiral ring of the gauge theory, but whether they are also the sufficient and appropriate degrees of freedom for a full low-energy effective description remains unknown. The precise reason for the appearance of the Veneziano-Yankielowicz superpotential $\sim S\log S - cS$ [6] remains somewhat mysterious as well, though in the matrix model it can be interpreted in terms of the measure. Another question is whether the matrix model approach, or a suitable modification thereof, is applicable to all possible gauge theories with all possible matter content. One may also wonder whether the integrable system that underlies the matrix model has any relation to the integrable system that underlies $N=2$ gauge theories [7, 8, 9].

In an attempt to shed light on some of these issues, we consider in this paper pure $\mathcal{N}=2$ theories with superpotential $\mathrm{Tr}\, W(\Phi)$ compactified on a circle with radius R. The integrable system that underlies the four-dimensional theory becomes much more prominent once the theory is compactified on a circle, and in addition the low-energy degrees of freedom are quite distinct. Therefore, we should obtain an interesting perspective on the matrix model results by compactifying the four-dimensional theories.

The compactification of the pure $\mathcal{N}=2$ $\mathrm{U}(N)$ theory was discussed in detail in [10]. Recall that in four dimensions there is a $2N$-dimensional moduli space, and at each point in the moduli space there is an auxiliary Riemann surface known as the Seiberg-Witten curve [11, 12, 13, 14] (for a review see e.g. [15, 16, 17]). The low-energy gauge couplings are given by the periods of the Seiberg-Witten curve, or in other words by the complex structure of the jacobian of the curve, which is a $2N$-dimensional torus. Once the theory is compactified, the moduli space becomes $4N$-dimensional. The extra moduli come from the four-dimensional gauge field. The component of the gauge field along the circle is a scalar from the three-dimensional point of view, and the remaining 3d gauge field can be dualized to a second scalar. Thus, each U(1) gauge field gives rise to two real scalars, that combine into one chiral superfield. The expectation values of these scalars provide the extra moduli. Since the low-energy gauge couplings were given by the complex structure of the jacobian, it should come as no surprise that the moduli space of the compactified theory is obtained by adding to each point in the moduli space of the uncompactified theory the jacobian of the associated Seiberg-Witten curve. Altogether the moduli space is a $4N$-dimensional hyperKähler manifold. According to [10], it has a distinguished complex structure that is independent of the radius R of the circle. Therefore, after including a superpotential, the holomorphic data (such as the value of the superpotential and the vacuum structure) will be independent of the radius R. This crucial fact shows that we should be able to recover the four-dimensional results directly for finite R, without the need to take the limit $R \to \infty$.

In several ways, life simplifies in the presence of a circle. First of all, the non-perturbative dynamics of the three-dimensional theories we consider here does not involve fractional instantons, but only regular 3d instantons (which are monopoles from the 4d point of view). Second of all, there are no degrees of freedom that become light anywhere on the moduli space. In four dimensions, there are massless monopoles and dyons on the moduli space, and these play a crucial role in the quantum physics. In the presence of a superpotential, some of these degrees of freedom condense, and this mechanism is responsible for the existence of supersymmetric vacua [11, 18, 19]. Once the compact circle is introduced, there are no such massless degrees of freedom for the theories we consider, and we therefore expect that the classical superpotential is actually identical to the quantum superpotential. The only issue is to find the right variables in which to express the superpotential. These variables are provided by the underlying integrable system, which in the case of $\mathrm{U}(N)$ is the periodic Toda chain, and whose phase space coincides with the moduli space of the unperturbed theory. Indeed, we will find that if the classical superpotential is $\mathrm{Tr}\, W(\Phi)$, the quantum superpotential is simply $\mathrm{Tr}\, W(M)$, with M the Lax matrix of the integrable system, if the order of W is not too large. This was first conjectured in [20], where it is also shown that this correctly produces several 4d vacua in the case of a mass deformation of the 4d $N = 4$ theory, and it was explained intuitively in [21]. The validity of this conjecture for massive (i.e. maximally confining) vacua for theories with higher order superpotentials was demonstrated in [22].

In this paper we will analyze the minima and other properties of the quantum superpotential $\mathrm{Tr}\, W(M)$ for arbitrary superpotentials W and gauge groups $\mathrm{U}(N)$. We will consider both massless and massive vacua. We will find that the vacuum structure is in complete agreement with the results obtained in four dimensions, in a series of examples. A general proof, using the integrable hierarchy of the periodic Toda chain, will be described in a

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separate publication [23]. Several features of the 4d theory have a simple interpretation on $R^3 \times S^1$. Massive vacua correspond to simple degenerate Lax matrices, lifting vacua from $\mathrm{U}(N)$ to $\mathrm{U}(tN)$ corresponds to applying a simple replica procedure to the Lax matrix, etc. We also discuss the semiclassical expansion on $R^3 \times S^1$, and the interpretation of the gluino condensate superfields. The combination $\sum_i S_i$ has a simple interpretation as a Lagrange multiplier in three dimensions, but the interpretation of the individual S_i remains unclear, and seems to involve a new version of mirror symmetry.

The outline of this paper is as follows. In section 2 we discuss some background material, such as general properties of $N = 2$ theories in three dimensions, and the vacuum structure in the absence and presence of a superpotential. In section 3 we describe the general conjecture and explain qualitatively why we expect it to be correct. In section 4 we discuss several examples and show that there is perfect agreement with the results obtained in four dimensions. In section 5 we discuss the semiclassical expansion, and in section 6 the relation with the integrable hierarchy of the periodic Toda chain. Supersymmetric minima determine a specific reduction of the hierarchy that is remarkably similar to the four-dimensional field theory expressions, and at the same time remarkably similar to the integrable system that underlies the Dijkgraaf-Vafa matrix model. In section 7 we discuss the interpretation of the gluino bilinear superfields S_i, and finally we present some puzzles and open problems.

2. Field theory background material

2.1 $\mathcal{N} = 2$ theories in $d = 4$

A pure $\mathcal{N} = 2$ theory with gauge group $\mathrm{U}(N)$ in four dimensions has as bosonic fields a scalar field Φ transforming in the adjoint representation which is part of an $\mathcal{N} = 1$ hypermultiplet, and a gauge field A_μ which is part of an $\mathcal{N} = 1$ vector multiplet. In a classical minimum of the scalar potential we can always diagonalize Φ,

$$\Phi = \begin{pmatrix} \phi_1 & & 0 \\ & \ddots & \\ 0 & & \phi_N \end{pmatrix} . \tag{2.1}$$

Such a value of Φ generically breaks the $\mathrm{U}(N)$ gauge group to $\mathrm{U}(1)^N$, but the Weyl group S_N also remains unbroken. Because of the action of the Weyl group, only symmetric polynomials in the ϕ_i are gauge invariant objects, and as a basis of these polynomials we can choose either $\mathrm{Tr}(\Phi^i)$ with $i = 1, \ldots, N$, or alternatively the coefficients in the characteristic polynomial,

$$\begin{aligned} P_N(x) &\equiv \det(x - \Phi) \\ &\equiv x^N + s_1 x^{N-1} + \ldots s_N \, . \end{aligned} \tag{2.2}$$

Classically, the moduli space is parametrized by the values of the s_i, and generically the gauge group is broken to $\mathrm{U}(1)^N$, but at special points some of the non-abelian gauge

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symmetry is restored. The quantum theory was first understood for SU(2) in the famous paper of Seiberg and Witten [11], and later generalized to and studied for many other gauge theories [12, 13, 14]. For a review see e.g. [15, 16, 17]. It turns out that the quantum moduli space is still parametrized by complex coordinates s_i, but there are no longer points where a non-abelian gauge symmetry is restored. Instead, there are special points in the moduli space where dyons and/or monopoles can become massless. For each point on the moduli space one can define an auxiliary curve, the Seiberg-Witten curve, given by

$$y^2 = P_N^2(x) - 4\Lambda^{2N} \tag{2.3}$$

whose periods determine the gauge couplings of the low-energy effective theory. The curve (2.3) describes a double cover of the complex x-plane, with Λ the dynamically generated scale of the $\mathcal{N} = 2$ theory. The classical limit corresponds to taking the limit $\Lambda \to 0$. The curve (2.3) has a jacobian T^{2N}, which is a complex torus with period matrix $\tau_{ij} = \int_{B_i} \omega_j$, where A_i, B_i is a standard basis of one-cycles on the curve, and ω_i form basis of holomorphic one-forms normalized so that $\int_{A_i} \omega_j = \delta_{ij}$. Massless monopoles and/or dyons appear whenever the curve (2.3) degenerates, i.e. some of the one-cycles collapse.

2.2 $\mathcal{N} = 1$ deformations in $d = 4$

Next, we consider what happens when we add a superpotential

$$\int d^4x d^2\theta \, \mathrm{Tr}\, W(\Phi) \tag{2.4}$$

with

$$W(\Phi) = \sum_{m=0}^{n} \frac{g_{m+1}}{m+1} \Phi^{m+1} \tag{2.5}$$

some polynomial of the superfield Φ. Classically, the equation

$$W'(\Phi) \equiv g_{n+1}(\Phi - a_1)(\Phi - a_2) \ldots (\Phi - a_n) = 0\,, \tag{2.6}$$

must hold and therefore each of the eigenvalues ϕ_i of Φ in equation (2.1) needs to be equal to one of the a_j. Thus the moduli space is reduced to a finite set of points, where at least a $\mathrm{U}(1)^N$ gauge symmetry remains unbroken. More precisely, if a_i is occupied N_i times, in other words N_i of the ϕ_j are equal to a_i, then the gauge symmetry is broken to

$$\mathrm{U}(N) \longrightarrow \mathrm{U}(N_1) \times \ldots \times \mathrm{U}(N_n)\,. \tag{2.7}$$

In the quantum theory, the pure $\mathcal{N} = 1$ theories with gauge group $\mathrm{U}(N_i)$ that appear classically confine, a nonzero gluino condensate $\langle \lambda\lambda \rangle \neq 0$ appears, and the gauge group is broken to U(1) (for $N_i > 0$). One can also show that necessarily some monopoles/dyons in the $\mathcal{N} = 2$ theory have to condense. Thus the breaking pattern of the gauge symmetry is

$$\mathrm{U}(N) \xrightarrow{\text{classical}} \mathrm{U}(N_1) \times \ldots \times \mathrm{U}(N_k) \xrightarrow{\text{quantum}} \mathrm{U}(1)^k \tag{2.8}$$

where k is the number of N_i that are not equal to zero.

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There are several approaches known in the literature to understanding the quantum theory. The first studies use Seiberg duality [24, 25]. The precise structure of the vacua can be obtained using a brane construction [19], one can use pure field theory methods [19, 26, 27], one can use Calabi-Yau geometry and geometric engineering [27], one can use matrix models [1, 2, 3] and one can use the generalized Konishi anomaly [5]. For our purposes, we will be mainly interested in a comparison to the vacuum structure as obtained using field theory methods. The field theory results can be summarized as follows:

1. First, express the quantities $\mathrm{Tr}(\Phi^{m+1})$ in terms of the s_i that appear in (2.2).

2. Next, determine the submanifold of the $\mathcal{N}=2$ theory on which there are at least $N-k$ mutually local massless monopoles/dyons. On this submanifold, the Seiberg-Witten curve degenerates and can be written in the form

$$y^2 = P_N^2(x) - 4\Lambda^{2N} = H_{N-k}^2(x)T_{2k}(x) \tag{2.9}$$

   for some polynomials H, T of degrees $N-k$, $2k$ respectively. As can be seen from (2.9), $N-k$ one-cycles have collapsed.

3. Minimize the classical superpotential, expressed in terms of the s_i, on this submanifold. The resulting extrema are the quantum vacua.

4. One can then show that the quantum vacua are in one-to-one correspondence with points on the moduli space where in addition to (2.9) we also have

$$G_{n-k}^2(x)T_{2k}(x) = W'(x)^2 + f_{n-1}(x) \tag{2.10}$$

   for some polynomials G, f of degrees $n-k, n-1$, and where T is the same polynomial that appears in (2.9). Equation (2.10) can be viewed as a degeneration of the matrix model curve.

5. Finally, we need to check that the classical limit of $P_N(x)$ is indeed $\prod_i (x-a_i)^{N_i}$, so that it indeed is a quantum vacuum corresponding to the appropriate classical vacuum.

For future reference, we also briefly summarize the matrix model approach of [3]. Their construction starts with the matrix integral

$$\int d\Phi e^{-\frac{1}{g^2}\mathrm{Tr}\, W(\Phi)}. \tag{2.11}$$

The symbol Φ now denotes an $M \times M$ matrix. Next we consider the planar diagrams in perturbation theory around a classical minimum where $\mathrm{U}(M) \to \mathrm{U}(M_1) \times \cdots \times \mathrm{U}(M_n)$. The free energy, that is the sum of the connected planar diagrams, is a function of the M_i and denoted by $\mathcal{F}(M_1, \ldots, M_n)$. Next, we replace gM_i by S_i to construct a function $\mathcal{F}(S_1, \ldots, S_n)$. With this definition of $\mathcal{F}$ the quantum superpotential for the $\mathcal{N}=1$

theory with classical superpotential W in a minimum where classically $\mathrm{U}(N)$ is broken to $\mathrm{U}(N_1) \times \ldots \times \mathrm{U}(N_n)$ is then

$$W_{\mathrm{eff}} = \sum_i N_i \frac{\partial \mathcal{F}}{\partial S_i} + \tau \sum_i S_i \,. \tag{2.12}$$

The S_i are superfields whose lowest components are the gaugino condensates $\mathrm{Tr}_{\mathrm{U}(N_i)}(\lambda\lambda)$. This quantum superpotential controls the complete chiral ring of the $\mathcal{N}=1$ theory and contains in that sense more information then the field theory result given above. The field theory results only describe the minima of (2.12) and are therefore recovered by minimizing (2.12) with respect to the S_i.

A subtlety in (2.12) is the inclusion of the Veneziano-Yankielowicz superpotential

$$W_{VY}^{(N_i)}(S_i) = S_i \left[\log \left(\frac{\Lambda^{3N_i}}{S_i^{N_i}} \right) + N_i \right] \tag{2.13}$$

for each of the classical unbroken gauge groups $\mathrm{U}(N_i)$. This can be attributed to the measure in the matrix model. For a pure $\mathcal{N}=1$ superpotential, (2.13) is the full quantum superpotential and its minima are at $S = e^{2\pi i t/N}\Lambda^3$ with $t = 0, \ldots, N-1$ and $W_{\mathrm{min}} = N\Lambda^3 e^{2\pi i t/N}$.

Having reviewed the situation in four dimensions, we now turn to compactifications to three dimensions.

2.3 $\mathcal{N}=4$ theories in $d=3$

Three-dimensional theories with $\mathcal{N}=4$ can be obtained by dimensionally reducing four-dimensional theories with $\mathcal{N}=2$. The structure of the Coulomb branch of such theories was studied in detail in [10], and they exhibit a rich set of physical phenomena such as mirror symmetry [28]. Under the dimensional reduction, the four-dimensional vector field A_μ decomposes in one scalar field r and a three dimensional vector field A_α. The three-dimensional vector can in turn be dualized to a second scalar via $\partial_\alpha \sigma = \epsilon_{\alpha\beta\gamma} F^{\beta\gamma}$. The two scalars combine into a complex scalar z, which is part of a 3d hypermultiplet. Thus, in three dimensions, the vector multiplet is dual to a hypermultiplet. This duality can be performed directly in the lagrangian for abelian gauge groups (see e.g. [29]), but not for non-abelian groups, similar to what happens with electric-magnetic duality in four dimensions. If we dimensionally reduce from four to three dimensions, σ will be a periodic variable, but r is unconstrained. If we instead compactify on a circle to go from four to three dimensions, both r and σ are periodic variables.

Classically, the complex variable z is $r+i\sigma$, and the action only depends on $Z+\bar{Z}$, where Z is the superfield with lowest component z. In perturbation theory the action remains a function of $Z+\bar{Z}$ only, but the relation between the vev of z, r and σ can become quite complicated. The periodicity of r and σ is therefore not always manifest in terms of z. For example, z could be a coordinate on a torus in Weierstrass form $y^2 = x^3 + ax + b$, but the fact that this is a torus is not manifest. Non-perturbatively, the action no longer needs to be a function of $Z+\bar{Z}$, as instantons generate a non-trivial dependence on the zero mode of σ.

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The compactification of a pure $\mathcal{N}=2$ theory on $R^3 \times S^1$ yields a three-dimensional theory with a moduli space which is parametrized by the vevs of ϕ and z. The gauge symmetry is broken everywhere to $\mathrm{U}(1)^N$, and both ϕ and z are diagonal. The moduli space is a hyperKähler manifold of dimension $4N$, and contains the moduli space of the four-dimensional theory that was parametrized only by ϕ. What we gain by going to three dimensions are the vevs of z, and these parametrize a $2N$-torus, which can be identified with the jacobian of the Seiberg-Witten curve. According to [10], one of the complex structures of the moduli space is independent of the radius R, and this complex structure is the one that will be relevant once we break to $\mathcal{N}=2$ in $d=3$. Since it is independent of R, the vacuum structure we find in three dimensions should be directly related to the vacuum structure in four dimensions.

2.4 $\mathcal{N}=2$ deformations in $d=3$

In this section we consider what happens when we add a superpotential

$$\int d^3x d^2\theta \, \mathrm{Tr}\, W(\Phi) \tag{2.14}$$

with

$$W(\Phi) = \sum_{m=0}^{n} \frac{g_{m+1}}{m+1} \Phi^{m+1} \,. \tag{2.15}$$

In four dimensions the gauge symmetry was broken according to (2.8), and since the complex structure of the moduli space did not depend on R, we expect that for every finite value of R this remains true. Therefore, we expect that the moduli space collapses to a finite collection of tori of dimension $2k$.

One of the main questions that we would like to answer in this paper is whether this is indeed true, and whether this can all be described using a suitable low-energy effective superpotential that depends on a suitable set of degrees of freedom. This can indeed be done, but to understand the result we first need to review some aspects of nonperturbative physics in three-dimensional gauge theories.

Non-perturbative physics in three dimensions is due to three-dimensional instantons, which from the four dimensional point of view are monopoles. They are classified by $\pi_2(\mathrm{U}(N)/\,\mathrm{U}(1)^N) = \mathbb{Z}^{N-1}$. Indeed, for each simple root there is a corresponding embedding $\mathrm{SU}(2) \subset \mathrm{U}(N)$, and for each such embedding there is a corresponding elementary monopole. A general monopole configuration is therefore labeled by a set of integers $\{n_1, \ldots, n_{N-1}\}$, counting the number of elementary monopole constituents. In a pure $\mathcal{N}=2$ theory in three dimensions, one can count the number of gaugino zero modes in a general monopole background using the Callias index theorem [30], and one finds that there are two zero modes only if one $n_i = 1$ and all other n_j vanish. We need two fermionic zero modes in order to get a non-trivial contribution to the superpotential, and therefore only the single elementary monopoles contribute to superpotential. Their contribution can be explicitly evaluated and the final result for the superpotential reads [31, 32, 33, 36]

$$W_{\text{quantum}} = e^{(Z_1 - Z_2)/g_3^2} + e^{(Z_2 - Z_3)/g_3^2} + \cdots + e^{(Z_{N-1} - Z_N)/g_3^2} \,. \tag{2.16}$$

Here, Z_i represents the diagonal entries of Z, and in the exponents one recognizes the simple roots of $\mathrm{U}(N)$; g_3 is the three-dimensional Yang-Mills coupling. The result (2.16) is exact, and shows runaway behavior. In a sense, (2.16) is less subtle than the Veneziano-Yankielowicz effective superpotential (2.13), because (2.16) involves a sum of ordinary instantons, and fractional instantons play no role. The situation in the presence of matter is quite a bit more subtle and is discussed in e.g. [29, 34].

2.5 $\mathcal{N} = 2$ deformations on $R^3 \times S^1$

Now we examine what happens if the $4d$ theory is put on $R^3 \times S^1$. In addition to the monopoles that contributed to (2.16), there is one more non-trivial gauge field configuration that contributes, which is the Kaluza-Klein monopole. This is present due to the existence of large gauge transformations along the S^1 [35].

The KK monopole adds one extra contribution to (2.16), and it becomes [10, 36]

$$W_{\text{quantum}} = e^{(Z_1-Z_2)/g_3^2} + \cdots + e^{(Z_{N-1}-Z_N)/g_3^2} + e^{-1/Rg_3^2} e^{(Z_N-Z_1)/g_3^2}. \tag{2.17}$$

The three-dimensional answer is recovered by taking $R \to 0$, whereas the decompactification limit is $R \to \infty$ while keeping the dynamical scale

$$\Lambda^{3N} \equiv e^{-1/g_4^2} \equiv e^{-1/Rg_3^2} \tag{2.18}$$

fixed.

In order to study the minima of (2.17), and in order to compare to the results we will find later, we first introduce a different set of variables

$$\begin{aligned} y_1 &= e^{(Z_1-Z_2)/g_3^2} \\ &\vdots \\ y_{N-1} &= e^{(Z_{N-1}-Z_N)/g_3^2} \\ y_0 &= e^{-1/Rg_3^2} e^{(Z_N-Z_1)/g_3^2}\,. \end{aligned} \tag{2.19}$$

The variables y_i are not unconstrained but obey

$$\prod_{i=0}^{N-1} y_i = \Lambda^{3N}. \tag{2.20}$$

To impose this constraint, we introduce a Lagrange multiplier field L, and with this field the superpotential (2.17) can be rewritten as

$$W_{\text{quantum}} = y_1 + \cdots + y_{N-1} + y_0 + L \log\left(\frac{\Lambda^{3N}}{\prod_{i=0}^{N-1} y_i}\right). \tag{2.21}$$

The minima of (2.21) are easily found, they are

$$y_{i,\min} = L_{\min} = \Lambda^3 e^{2\pi i t/N}\,, \qquad W_{\min} = N\Lambda^3 e^{2\pi i t/N}\,. \tag{2.22}$$

Indeed, the dependence on R has dropped out of (2.21) and (2.22), and the results are identical to the results obtained from the Veneziano-Yankielowicz superpotential (2.13).

Before discussing the general case, we discuss the simplest deformation of an $\mathcal{N}=4$ theory to a $\mathcal{N}=2$ theory, namely by a mass term $W(\Phi)=\frac{1}{2}m\Phi^2$. In the presence of such a mass term, there is a pure $\mathcal{N}=2$ theory at low energies, whose scale $\tilde{\Lambda}$ is related to the high energy $\mathcal{N}=4$ scale Λ via scale matching as

$$\tilde{\Lambda^3} = m\Lambda^2\,. \tag{2.23}$$

If we substitute this in (2.21), and redefine $y_i \to m y_i$, the superpotential becomes

$$W = m\sum_{i=0}^{N-1} y_i + L\log\left(\frac{\Lambda^{2N}}{\prod_{i=0}^{N-1} y_i}\right). \tag{2.24}$$

The extrema of this superpotential are as before, we merely made a change of variables. The superpotential (2.24) only contains gauge degrees of freedom, but we could also have chosen to include the diagonal entries of Φ in the superpotential. They would simply appear through an extra mass term,

$$W = m\sum_{i=0}^{N-1} y_i + L\log\left(\frac{\Lambda^{2N}}{\prod_{i=0}^{N-1} y_i}\right) + \frac{1}{2}m\sum_{i=0}^{N-1}\phi_i^2\,. \tag{2.25}$$

The minima of (2.25) are the same as before, as $\phi_i=0$ at the extremum. As we will see, (2.25) is literally the superpotential we obtain if we evaluate $\frac{1}{2}m\,\mathrm{Tr}\,M^2$, with M the Lax matrix of the periodic Toda chain. In general, we will find a superpotential which is a function of y_i and ϕ_i, where the y_i are subject to $\prod y_i = \Lambda^{2N}$.

3. The proposal

We now turn to deformations involving a general superpotential $\int d^4x d^2\theta\,\mathrm{Tr}\,W(\Phi)$ on $R^3\times S^1$. As explained in the introduction, since there are no new massless degrees of freedom on the moduli space, all we need to do is to figure out how to write the operators $\mathrm{Tr}(\Phi^m)$ in terms of suitable holomorphic variables on the $4N$-dimensional hyperKähler moduli space of the unperturbed theory. The resulting expression should be the low-energy effective quantum superpotential.

To find these variables we need to use the fact that the moduli space is at the same time the phase space of a (complexified) integrable system. This integrable system is the periodic Toda chain, and its relation to the Seiberg-Witten curve and $\mathcal{N}=2$ theories in four dimensions was found in [7, 8, 9]. The periodic Toda chain is described by the hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \Lambda^2 e^{q_1-q_2} + \cdots + \Lambda^2 e^{q_{N-1}-q_N} + \Lambda^2 e^{q_N-q_1} \tag{3.1}$$

with coordinates q_i and momenta p_i, and we introduced a parameter Λ^2 which will later be identified with a field theory scale. Notice the similarity of this expression to that in (2.25). The dynamics described by the hamiltonian (3.1) is integrable: there exists a Lax operator $M(p_i,q_i)$, which is an $N\times N$ matrix and a function of coordinates and momenta, such that the time evolution given by the hamiltonian (3.1) can equivalently be described by the equation

$$\frac{\partial M}{\partial t} = [M, \mathcal{L}(M)] \,. \tag{3.2}$$

The linear operator $\mathcal{L}$ will be described in more detail in section 6. One of the implications of (3.2) is that the variables $u_k = \frac{1}{k+1}\operatorname{Tr}(M^k)$ are constants of the equations of motion. With further work [37] one can also show that they Poisson commute. They provide a complete set of action variables and generate commuting flows

$$\frac{\partial M}{\partial t_k} = \{M, u_k\} = \left[M, \mathcal{L}\left(\frac{1}{k+1}M^k\right)\right] \tag{3.3}$$

on the phase space. Thus, the phase space admits action-angle variables, and the angle variables are linear in the variables t_k. The t_k provide a local set of coordinates on the jacobian of the Seiberg-Witten curve, and are closely related to gauge transformations (recall that the angular coordinates that describe the jacobian were obtained by dualizing the gauge field). Therefore, it is in many ways natural to identify the gauge invariant quantities $\frac{1}{k+1}\operatorname{Tr}(\Phi^{k+1})$ with the conserved quantities $u_k = \frac{1}{k+1}\operatorname{Tr}(M^k)$ of the dynamical system. This is also what the original discussion of the role of the integrable system in $\mathcal{N} = 2$ theories in four dimensions implies [7, 8]. Therefore, we conjecture, following [20, 21, 22] that the quantum superpotential can be obtained from the classical one by the rule

$$\int d^4x d^2\theta \operatorname{Tr} W(\Phi) \longrightarrow \int d^4x d^2\theta \operatorname{Tr} W(M) \,, \tag{3.4}$$

with M the Lax matrix of the integrable system. In order to compare this to the field theory discussion in section 2, we need to explain the relation between the coordinates and momenta in (3.1) and the field theory degrees of freedom that appeared in our discussion in section 2. That can be done by comparing the hamiltonian (3.1) to the superpotential (2.25). According to (3.4), these two should be identified with each other. Therefore, the momenta p_i of the integrable system should naturally be identified with the eigenvalues ϕ_i of the adjoint superfield Φ as in (2.1). Furthermore, the variables y_i can be identified with $\Lambda^2 e^{q_i - q_{i+1}}$, where the indices on q are identified modulo N.

To make the statement in (3.4) more explicit, we need to give the explicit form of the Lax matrix. It is given by [38]

$$M = \begin{pmatrix} p_1 & \Lambda^2 e^{q_1-q_2} & 0 & \dots & z \\ 1 & p_2 & \Lambda^2 e^{q_2-q_3} & \dots & 0 \\ 0 & 1 & p_3 & \dots & 0 \\ . & . & . & \dots & \Lambda^2 e^{q_{N-1}-q_N} \\ e^{q_N-q_1} z^{-1} & . & . & 1 & p_N \end{pmatrix} \equiv \begin{pmatrix} \phi_1 & y_1 & 0 & \dots & z \\ 1 & \phi_2 & y_2 & \dots & 0 \\ 0 & 1 & \phi_3 & \dots & 0 \\ . & . & . & \dots & y_{N-1} \\ y_0 z^{-1} & . & . & 1 & \phi_N \end{pmatrix} . \tag{3.5}$$

The parameter z is a so-called spectral parameter. As long as the superpotential does not contain higher powers than $N-1$, the superpotential is independent of z. If we compute $\operatorname{Tr}(M^N)$, we find a z-dependent constant that we simply drop. Higher powers can also be taken into account, but we defer that discussion to section 6.

Our final proposal is therefore that the quantum superpotential is given by (3.4), with M given by (3.5). The y_i are constrained to satisfy $\prod y_i = \Lambda^{2N}$, but we will find it convenient to keep the y_i unconstrained in the Lax matrix M, and to impose the constraint via a Lagrange multiplier term

$$L \log\left(\frac{\Lambda^{2N}}{\prod_{i=0}^{N-1} y_i}\right) \tag{3.6}$$

in the superpotential.

The spectral curve associated to the Lax matrix M is defined by the equation

$$\det(x - M) \equiv P_N(x) + (-1)^N (z + \Lambda^{2N} z^{-1}) = 0\,. \tag{3.7}$$

If we introduce a new variable $y = 2z + (-1)^N P_N(x)$, the spectral curve becomes

$$y^2 = P_N(x)^2 - 4\Lambda^{2N} \tag{3.8}$$

which is exactly the same as the Seiberg-Witten curve (2.3). Therefore, the quantities $\mathrm{Tr}(M^{k+1})$ provide coordinates on the moduli space of the $\mathcal{N} = 2$ theory in four dimensions, and this provides further motivation for the proposal (3.4).

3.1 Features of the proposal

Before working our way through a list of examples, we mention some general features of the superpotential (3.4).

First of all, it is interesting to observe that the quantum superpotential is simply obtained by substituting the Lax matrix in the classical superpotential. No matrix models need to be solved. Perhaps this is a manifestation of the observation that the superpotential of a d dimensional gauge theory appears to require solving a $d-4$ dimensional auxiliary theory, as suggested in [39]. A -1-dimensional theory indeed requires no integrations whatsoever.

Another appealing feature of (3.4) is that it is polynomial in y_i, and therefore the expansion in powers of y corresponds to a monopole expansion. In the classical limit where $\Lambda \to 0$, all y_i should also be taken to zero, and all non-perturbative effects disappear. What is left is the classical superpotential expressed as a function of the ϕ_i.

In the periodic Toda chain, the center of mass coordinate $\sum q_i$ does not appear, and as a consequence the variables y_i are not independent but satisfy $\prod_i y_i = \Lambda^{2N}$. This center of mass coordinate corresponds to the U(1) factor in $\mathrm{U}(N) = \mathrm{U}(1) \times \mathrm{SU}(N)$, and this U(1) is always left unbroken. It accordingly never appears in the superpotential. The remaining $\mathrm{U}(1)^{k-1}$ gauge symmetry that is left unbroken (see (2.8)) will emerge as $k-1$ free parameters that are left after extremizing the superpotential. In other words, the superpotential will not have isolated vacua in general, but will have moduli spaces of vacua of real dimension $2(k-1)$. It is easy to see that the flows of the integrable system have to map extrema of the superpotential into extrema. Therefore what should happen is that $N-k$ of the flows leave the extremum invariant, while the remaining $k-1$ flows generate the $k-1$ free complex parameters dual to the unbroken $\mathrm{U}(1)^{k-1}$. The gauge couplings of this unbroken $\mathrm{U}(1)^{k-1}$ can be extracted from the spectral curve, by studying the complex structure of the jacobian of the spectral curve at the extrema.

The above picture is based on a comparison to the known results in four dimensions (see section 2.2), but we still need to show that this is indeed what happens. Intuitively, what happens is the following. The superpotential is a linear combination of action variables, and correspondingly generates a flow on the moduli space. In order for the superpotential to have an extremum, this flow needs to have a stationary point. Since all flows are linear motions on the jacobian of the associated curve, this can only happen if the jacobian and Seiberg-Witten curve degenerate. What is not clear is why the existence of a stationary point of a single flow should automatically imply that in fact $N-k$ flows are stationary at this point. Correspondingly, $N-k$ one-cycles of the Seiberg-Witten curve should collapse, and the jacobian should degenerate to a $(k-1)$-dimensional complex torus.

The way in this happens is somewhat mysterious, and is best explained in terms of the integrable Toda system. The relevant equations are summarized in section 6, but we defer a full proof of these statements to a separate publication [23].

The Lagrange multiplier we introduced in (3.6) is not just there for technical convenience. It turns out that the expectation value of L in an extremum is identical to the value of the $\mathrm{U}(N)$ gluino bilinear superfield S in the corresponding extremum in the four-dimensional theory. This will be further discussed in section 7.

A last feature we would like to point out is that the Lax matrix is not invariant under general permutations, i.e. under the Weyl group of $\mathrm{U}(N)$. It is only form invariant under a $\mathbb{Z}_N$ subgroup of cyclic permutations of the ϕ_i and the y_i. A choice of classical vacuum configuration corresponds to choosing each ϕ_i equal to some a_j (the solution of $W'(x)=0$, see (2.6)), and all $y_i=0$. It is not guaranteed that all choices of ϕ_i have a corresponding minimum of the quantum superpotential. Because permutation symmetry is broken, it could be that one needs to order the ϕ_i in a suitable way in order to find a minimum of the quantum theory. We will see that this is indeed what happens. Precisely how one should order the eigenvalues in general is an interesting open problem that we have not yet been able to solve.

4. Examples

In this section we discuss in detail how our proposal works in various examples. For each of the classical vacua that gives rise to a symmetry breaking pattern as in (2.8), we expect to find $\prod_{N_i>0} N_i$ vacua, as each $\mathrm{U}(N_i)$ low energy gauge group has N_i distinct quantum vacua. The various choices of vacua show up as choices of phases in the solutions below. We will compare the results to the four dimensional results as described in section 2.2.

4.1 U(2)

The first example that we will discuss is U(2) with tree level superpotential

$$W=g_1\,\mathrm{tr}(\Phi)+\frac{g_2}{2}\,\mathrm{tr}(\Phi^2)+\frac{g_3}{3}\,\mathrm{tr}(\Phi^3)\,. \tag{4.1}$$

The Lax operator in this case reads

$$M = \begin{pmatrix} \phi_1 & y_1 + z \\ 1 + y_0 z^{-1} & \phi_2 \end{pmatrix} . \tag{4.2}$$

The polynomial $P(x)$ appearing in the spectral curve equals

$$P(x) = (x - \phi_1)(x - \phi_2) - y_0 - y_1 . \tag{4.3}$$

The resulting effective superpotential, using (3.4), reads

$$W_{\text{eff}} = W(\phi_1) + W(\phi_2) + (y_0 + y_1)(g_2 + g_3(\phi_1 + \phi_2)) + L \log\left(\frac{\Lambda^4}{y_0 y_1}\right) . \tag{4.4}$$

The equations for the extrema of W_{eff} are

$$\begin{aligned} W'(\phi_1) + g_3(y_0 + y_1) = 0 \qquad & g_2 + g_3(\phi_1 + \phi_2) = \frac{L}{y_0} \\ W'(\phi_2) + g_3(y_0 + y_1) = 0 \qquad & g_2 + g_3(\phi_1 + \phi_2) = \frac{L}{y_1} \\ y_0 y_1 = \Lambda^4 . & \end{aligned} \tag{4.5}$$

We now consider the various possible solutions of these equations.

Case 1: $L \neq 0$. This necessarily implies that $y_0 = y_1 = \epsilon\Lambda^2$ with $\epsilon^2 = 1$. Next, we observe that the difference between the first two equations can be rewritten as

$$(\phi_1 - \phi_2)(g_2 + g_3(\phi_1 + \phi_2)) = 0 . \tag{4.6}$$

Since $L \neq 0$ and y_0 and y_1 can never be zero, the second factor in (4.6) cannot vanish, and therefore $\phi_1 = \phi_2$. The full solution is therefore the following. Define ϕ_0 to be any root of the equation

$$W'(\phi_0) + 2g_3\epsilon\Lambda^2 = 0 , \tag{4.7}$$

then

$$\phi_1 = \phi_2 = \phi_0 , \qquad y_0 = y_1 = \epsilon\Lambda^2 , \qquad L = 2\epsilon\Lambda^2(g_2 + 2g_3\phi_0) . \tag{4.8}$$

By taking the classical limit $\Lambda \to 0$, we see that ϕ_1 and ϕ_2 are both the same solution of the equation $W'(\phi) = 0$. Thus, this solution describes the maximally confining case with one solution of $W'(\phi) = 0$ doubly occupied, and the other not occupied. Classically the gauge symmetry is still U(2), but it is broken quantum mechanically to U(1). There are two different solutions depending on the choice of ϵ.

Let us compare this to the expected 4d field theory answer. According to the recipe in section 2.2, we first need to parametrize the locus where $P(x)^2 - 4\Lambda^4$ has a double zero, at say $x = x_0$. With $P(x) = x^2 + s_1 x + s_2$ we need that $P'(x_0) = 0$ and $P(x_0) = -2\eta\Lambda^2$ where $\eta^2 = 1$. This implies that $s_1 = -2x_0$ and $s_2 = x_0^2 - 2\eta\Lambda^2$. The effective superpotential, expressed in terms of s_1 and s_2, is equal to

$$W_{\text{eff}} = -g_1 s_1 + \frac{g_2}{2}\left(s_1^2 - 2s_2\right) + \frac{g_3}{3}\left(-s_1^3 + 3s_1 s_2\right) \tag{4.9}$$

and when evaluated on the locus where $P(x)^2 - 4\Lambda^4$ has a double zero at $x = x_0$ it becomes

$$W_{\rm eff} = 2g_1 x_0 + g_2\left(x_0^2 + 2\eta\Lambda^2\right) + \frac{2g_3}{3}\left(x_0^3 + 6\eta\Lambda^2 x_0\right). \tag{4.10}$$

Extremizing this with respect to x_0 yields

$$2\left(W'(x_0) + 2g_3\eta\Lambda^2\right) = 0\,, \tag{4.11}$$

and for each solution x_0 the polynomial $P(x) = x^2 + s_1 x + s_2$ reduces to

$$P(x) = (x - x_0)^2 - 2\eta\Lambda^2\,. \tag{4.12}$$

This is exactly the same polynomial as the one that we obtain from (4.3) by substituting (4.8), with $\eta = \epsilon$. Thus, in the maximally confining case we reproduce the field theory answer.

Case 2: $L = 0$**.** It is straightforward to find the solution in this case. We can take for example ϕ_1 to be completely arbitrary, so that $\phi_2 = -\phi_1 - g_2/g_3$. In addition, y_0 and y_1 are the two roots of the equation

$$y^2 + \frac{1}{g_3}W'(\phi_1)y + \Lambda^4 = 0\,. \tag{4.13}$$

To understand the classical limit we send $\Lambda \to 0$ and also take $y_i \to 0$. In that limit, ϕ_1 and ϕ_2 become the two distinct roots of the equation $W'(\phi) = 0$. Thus, this is the case where the gauge symmetry is classically broken to $\mathrm{U}(1)\times\mathrm{U}(1)$. In contrast to the previous situation, the quantum theory has a flat direction, parametrized by ϕ_1. As discussed in sections 2.4 and 3.1, this flat direction represents the fact that at the quantum level there remains a $\mathrm{U}(1)\times\mathrm{U}(1)$ gauge symmetry. The first, diagonal $\mathrm{U}(1)$ is completely decoupled and never present in the superpotential, the second $\mathrm{U}(1)$ is parametrized by ϕ_1. Equivalently, we can parametrize it by y_0/y_1, which is the combination that also represented this $\mathrm{U}(1)$ classically. The polynomial (4.3) is equal to

$$P(x) = x^2 + \frac{g_2}{g_3}x + \frac{g_1}{g_3} = \frac{W'(x)}{g_3}\,. \tag{4.14}$$

As expected, the $P(x)$ does not depend on the free parameter ϕ_1 at all. According to (2.9) and (2.10) we get the same result in four dimensions. Therefore, also in this case we find complete agreement with 4d field theory.

In summary, the structure of the vacua in our proposal for three dimensions matches exactly the results in four dimensions, at least for $\mathrm{U}(2)$ with a cubic superpotential.

4.2 U(3)

Our next example is U(3), again with tree level superpotential

$$W = g_1\,\mathrm{tr}(\Phi) + \frac{g_2}{2}\,\mathrm{tr}(\Phi^2) + \frac{g_3}{3}\,\mathrm{tr}(\Phi^3)\,. \tag{4.15}$$

The Lax operator is

$$M = \begin{pmatrix} \phi_1 & y_1 & z \\ 1 & \phi_2 & y_2 \\ y_0 z^{-1} & 1 & \phi_3 \end{pmatrix} . \tag{4.16}$$

The polynomial $P(x)$ appearing in the spectral curve equals

$$P(x) = (x - \phi_1)(x - \phi_2)(x - \phi_3) - (y_0 + y_1 + y_2)x + (y_2\phi_1 + y_0\phi_2 + y_1\phi_3) . \tag{4.17}$$

The resulting effective superpotential, again using (3.4), becomes

$$\begin{aligned} W_{\text{eff}} &= W(\phi_1) + W(\phi_2) + W(\phi_3) + g_2(y_0 + y_1 + y_2) + \\ &\quad + g_3(y_0(\phi_1 + \phi_3) + y_1(\phi_1 + \phi_2) + y_2(\phi_2 + \phi_3)) + L \log\left(\frac{\Lambda^6}{y_0 y_1 y_2}\right) . \end{aligned} \tag{4.18}$$

The equations for the extrema of W_{eff} are

$$\begin{aligned} W'(\phi_1) + g_3(y_0 + y_1) = 0 \qquad & g_2 + g_3(\phi_1 + \phi_2) = \frac{L}{y_1} \\ W'(\phi_2) + g_3(y_1 + y_2) = 0 \qquad & g_2 + g_3(\phi_2 + \phi_3) = \frac{L}{y_2} \\ W'(\phi_3) + g_3(y_2 + y_0) = 0 \qquad & g_2 + g_3(\phi_3 + \phi_1) = \frac{L}{y_0} \\ y_0 y_1 y_2 = \Lambda^6 . & \end{aligned} \tag{4.19}$$

The first three equations can be used to solve for y_0, y_1 and y_2. When we substitute this in the next three equations, they become

$$\begin{aligned} (-W'(\phi_1) + W'(\phi_2) - W'(\phi_3))(g_2 + g_3(\phi_3 + \phi_1)) &= 2g_3 L \\ (-W'(\phi_1) - W'(\phi_2) + W'(\phi_3))(g_2 + g_3(\phi_1 + \phi_2)) &= 2g_3 L \\ (W'(\phi_1) - W'(\phi_2) - W'(\phi_3))(g_2 + g_3(\phi_2 + \phi_3)) &= 2g_3 L . \end{aligned} \tag{4.20}$$

An interesting simplification appears if we consider the differences of pairs of these equations, which take the form

$$(\phi_i - \phi_j)(\phi_1 + \phi_2 + \phi_3 - 2a_1 - a_2)(\phi_1 + \phi_2 + \phi_3 - a_1 - 2a_2) = 0 \tag{4.21}$$

for some i, j, and where a_1, a_2 are the two extrema of W',

$$W'(\phi) = g_3(\phi - a_1)(\phi - a_2) . \tag{4.22}$$

Again, we distinguish two cases. Either all ϕ_i are equal, or at least two of them are different.

Case 1: all ϕ_i equal. This case is straightforward to analyze. All y_i are identical and equal to $\omega\Lambda^2$, with ω some third root of unity. All ϕ_i are also equal and a solution of the equation

$$W'(\phi_i) + 2\omega g_3 \Lambda^3 = 0\,. \tag{4.23}$$

The polynomial in (4.17) becomes

$$P(x) = (x-\phi_1)^3 - 3\omega\Lambda^2(x-\phi_1) \tag{4.24}$$

with ϕ_1 a solution of (4.23). For comparison with the field theory answer, we also compute $P(x)^2 - 4\Lambda^6$, which equals

$$P(x)^2 - 4\Lambda^6 = (x-\phi_1-\omega^{\frac{1}{2}}\Lambda)^2(x-\phi_1+\omega^{\frac{1}{2}}\Lambda)^2(x-\phi_1-2\omega^{\frac{1}{2}}\Lambda)(x-\phi_1+2\omega^{\frac{1}{2}}\Lambda)\,. \tag{4.25}$$

This has the expected form for the maximally confining case. Indeed, in the classical limit all ϕ_i reduce to either a_1 or a_2, so this situation is the one where classically the gauge group remains unbroken. In the quantum theory the gauge symmetry is broken to U(1), and there are three different vacua depending on the choice of ω.

Let us briefly check that we get the same answer in field theory. First, we need to parametrize s_1, s_2 and s_3 in such a way that $P(x)^2 - 4\Lambda^6$ has two double zeroes. This parametrization is easily found to be

$$s_1 = -3t\,, \qquad s_2 = 3t^2 - 3\omega\Lambda^2\,, \qquad s_3 = -t^3 + 3\omega\Lambda^2 t\,. \tag{4.26}$$

The effective superpotential, restricted to the locus (4.26) parametrized by t is

$$W_{\rm eff}(t) = 3W(t) + 3g_2\omega\Lambda^2 + 6g_3\omega\Lambda^2 t \tag{4.27}$$

which is extremal for $W'(t) + 2g_3\omega\Lambda^2 = 0$. With this value for t, we can insert (4.26) in (4.17) to compute $P(x)$, and we immediately see it is the same as (4.24).

Both for U(2) as well as U(3) the maximally confining case had all ϕ_i equal, and all y_i equal as well. This continues to be true for U(N), as we will elaborate on in section 4.3.

Case 2: some ϕ_i different: in this case (4.21) implies that

$$\phi_1 + \phi_2 + \phi_3 = 2a_1 + a_2\,. \tag{4.28}$$

We could also have chosen the left hand side to be equal to $a_1 + 2a_2$, but we can then simply exchange a_1 and a_2, so there is no loss in generality in taking (4.28). The single linear relation (4.28) collapses the three equations in (4.20) into a single equation, since all differences vanish. We can choose to eliminate ϕ_3 from the equation using (4.28). We are then left with a single equation from (4.20), plus the additional equation $y_0 y_1 y_2 = \Lambda^6$. Those are two equations for the three variables L, ϕ_1, ϕ_2, and therefore there will be a free parameter in the solution. This is similar to what we saw for U(2) when it is classically broken to U(1) $\times$ U(1). Here, as is clear from (4.28), we are considering the situation where U(3) is classically broken to U(1) $\times$ U(2). In the quantum theory it is further broken to U(1) $\times$ U(1). One of these U(1)'s is the trivial diagonal U(1), the other is the free parameter that we find here.

The full solution can be parametrized as follows

$$\begin{aligned}
\phi_1 &= a_1 + \tilde{\phi}_1 & y_0 &= \frac{\epsilon\Lambda^3}{\tilde{\phi}_2} \\
\phi_2 &= a_1 + \tilde{\phi}_2 & y_1 &= \tilde{\phi}_1\tilde{\phi}_2 \\
\phi_3 &= a_2 - \tilde{\phi}_1 - \tilde{\phi}_2 & y_2 &= \frac{\epsilon\Lambda^3}{\tilde{\phi}_1} \\
L &= -\epsilon g_3 \Lambda^3
\end{aligned} \tag{4.29}$$

with $\epsilon^2 = 1$ and $\tilde{\phi}_1, \tilde{\phi}_2$ subject to one constraint

$$\epsilon\Lambda^3 + \tilde{\phi}_1\tilde{\phi}_2(a_1 - a_2 + \tilde{\phi}_1 + \tilde{\phi}_2) = 0\,. \tag{4.30}$$

To compare with field theory we note that (4.29) substituted in (4.17) yields

$$P(x) = (x - a_1)^2(x - a_2) + 2\epsilon\Lambda^3 \tag{4.31}$$

from which we deduce

$$P(x)^2 - 4\Lambda^6 = (x - a_1)^2\left(\frac{1}{g_3^2}W'(x)^2 + 4\epsilon\Lambda^3(x - a_2)\right). \tag{4.32}$$

This is exactly the form that one would get from a field theory analysis (see section 2.2), and therefore the result agrees with the four-dimensional field theory expectation. Notice that the two solutions of $\epsilon^2 = 1$ correspond to two inequivalent quantum vacua, as expected for $\mathrm{U}(3) \to \mathrm{U}(1) \times \mathrm{U}(2)$.

Thus, for U(3) we also reproduce the complete vacuum structure that we expect from four dimensions.

4.3 U(N) with quadratic superpotential

The analysis for a U(N) theory with a quadratic superpotential,

$$W = g_1\,\mathrm{tr}(\Phi) + \frac{g_2}{2}\,\mathrm{tr}(\Phi^2) + L\log\left(\frac{\Lambda^{2N}}{y_0\cdots y_{N-1}}\right) \tag{4.33}$$

can be done for general N. Substituting the Lax matrix for the adjoint scalar Φ into the superpotential gives the following effective superpotential

$$W_{\mathrm{eff}} = \sum_{i=0}^{N-1}(g_2 y_i + g_1\phi_{i+1} + \tfrac{1}{2}g_2\phi_{i+1}^2) + L\log\left(\frac{\Lambda^{2N}}{y_0\cdots y_{N-1}}\right)$$

with the equations for the extrema

$$\begin{gathered}
y_0\ldots y_{N-1} = \Lambda^{2N} \\
g_2 - \frac{L}{y_i} = 0 \Rightarrow y_i = \frac{L}{g_2} \\
g_2\phi_i + g_1 = 0\,.
\end{gathered}$$

From the second equation we can learn that the y's are all equal, this enables us to solve for L using the first equation

$$L = \epsilon g_2 \Lambda^2 , \qquad \epsilon^N = 1 . \tag{4.34}$$

Substituting L into the equation for y_i yields

$$y_i = y = \Lambda^2 \epsilon .$$

Further, we also see that all the ϕ's occupy the root $x = -g_1/g_2$ of $W'(x)$

$$W'(x) = g_2 \left(x + \frac{g_1}{g_2} \right) .$$

The ϕ's should always occupy the roots of $W'(x)$ classically ($\Lambda \to 0$), but here it is even true in the quantum case.

The superpotential in the extrema is then

$$W_{\rm extr} = N \left(y - \frac{g_1^2}{2g_2} \right) = N \left(\Lambda^2 \mathrm{e}^{\frac{2\pi i l}{N}} - \frac{g_1^2}{2g_2} \right) .$$

To compare with field theory results we compute the characteristic polynomial $P_N(x) = \det(x 1_N - \Phi)$. According to appendix A this is either a Chebyshev polynomial of the first or of the second kind. By evaluating $P_1(x)$ and $P_2(x)$ one can see that one has to pick the polynomials of the first kind

$$P_N(x) = 2 y^{N/2} T_N \left(\frac{x + \frac{g_1}{g_2}}{2\sqrt{y}} \right) .$$

These $P_N(x)$ are in perfect agreement with field theory results

$$\begin{aligned} P_N(x)^2 - 4\Lambda^{2N} &= 4\Lambda^{2N} \left(1 - T_N^2 \left(\frac{x + \frac{g_1}{g_2}}{2\sqrt{y}} \right) \right) = 4\Lambda^{2N} \left(\left(\frac{x + \frac{g_1}{g_2}}{2\sqrt{y}} \right)^2 - 1 \right) U_{N-1}^2 \left(\frac{x + \frac{g_1}{g_2}}{2\sqrt{y}} \right) \\ &= \Lambda^{2(N-1)} \epsilon^{N-1} \left(\tfrac{1}{g_2^2} (W')^2 - 4\Lambda^2 \epsilon \right) U_{N-1}^2 \left(\frac{x + \frac{g_1}{g_2}}{2\sqrt{y}} \right) . \end{aligned}$$

4.4 Maximally confining vacua

In the previous sections we found all vacua for some simple potentials and small rank of $\mathrm{U}(N)$. In this section we consider the simple vacua with ϕ_i taken to be equal. These are considered for arbitrary N and for polynomial potentials of degree $n \leq N+1$. These vacua are maximally confining.

The maximally confining vacua of a $\mathrm{U}(N)$ theory with superpotential

$$W(x) = \sum_{k=0}^{n} \frac{g_{k+1}}{k+1} x^{k+1} \tag{4.35}$$

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are obtained by taking all $y_i = (\eta\Lambda)^2$ with $\eta^{2N} = 1$, and taking all $\phi_i = \phi$, where ϕ is such that tr $W'(M) = 0$. Because of the tridiagonal form of M this requirement on ϕ is equivalent to the condition $c_0 = 0$ for the coefficient in the Laurent expansion

$$W\left(\phi + \xi + \frac{y}{\xi}\right) = \sum_{k=-n}^{n} c_k \xi^k \,. \tag{4.36}$$

Expressed in terms of the coefficients of the superpotential, the condition $c_0 = 0$ reads

$$\sum_{k=0}^{n} g_{k+1} \left(\sum_{l=0}^{\lfloor k/2 \rfloor} \frac{k!}{(l!)^2 (k-2l)!} \phi^{k-2l} y^l \right) = 0 \,. \tag{4.37}$$

with $\lfloor k \rfloor := \max\{m \in \mathbb{Z} | m \leq k\}$. This can be expressed in terms of a hypergeometric function

$$\sum_{k=0}^{n} g_{k+1} \phi^k \, {}_2F_1 \left(-\frac{k}{2}, -\frac{k}{2} + \frac{1}{2}; 1; \frac{4y}{\phi^2} \right) = 0 \,. \tag{4.38}$$

On the other hand, according to (2.9) and (2.10) the factorization of the polynomial $P_N(x)$ appearing in the spectral curve imposes an apparently different condition on ϕ. With all ϕ_i equal, $P_N^2(x) - 4\Lambda^{2N}$ must have $N-1$ double zeroes:

$$P_N^2(x) - 4\Lambda^{2N} = T(x) H_{N-1}^2(x) \,. \tag{4.39}$$

In this case P_N is a Chebyshev polynomial: $P_N(x) = 2(\eta\Lambda)^N T_N\left(\frac{x-\phi}{2\eta\Lambda}\right)$, with $\eta^{2N} = 1$. Together with the convenient definition of the Chebyshev polynomials of first and second kind

$$\begin{aligned} T_n(\cos x) &= \cos nx \\ U_n(\cos x) &= \frac{\sin(n+1)x}{\sin x} \end{aligned} \tag{4.40}$$

this factorization is directly observed

$$\frac{P_N^2(x) - 4\Lambda^{2N}}{4\Lambda^{2N}} = \left(T_N^2 \left(\frac{x-\phi}{2\eta\Lambda} \right) - 1 \right) = \left[\left(\frac{x-\phi}{2\eta\Lambda} \right)^2 - 1 \right] U_{N-1}^2 \left(\frac{x-\phi}{2\eta\Lambda} \right) . \tag{4.41}$$

So after rescaling one obtains

$$T(x) = x^2 - 2\phi x + \phi^2 - 4y \tag{4.42}$$

for any maximally confining vacuum, irrespective of the rank of the gauge group.

From the point of view of four dimensional gauge theory, it is expected that $T(x)$ must be related to the superpotential as

$$T(x)\, G_{n-1}^2(x) = (W'(x))^2 + f_{n-1}(x) \tag{4.43}$$

for some polynomials $G_{n-1}(x)$, $f_{n-1}(x)$ of degree $n-1$. Equation (4.43) can be seen as a set of $2n+1$ conditions that the coefficients of each power of x match. Of these conditions

$2n$ can be satisfied by appropriate choice of coefficients of $G_{n-1}(x)$ and $f_{n-1}(x)$. The final remaining condition turns out to be $c_0 = 0$, in (4.36), which relates ϕ and y to the superpotential at criticality.

To see this, examine the conditions that $G_{n-1}(x)$ and $f_{n-1}(x)$ can be found such that (4.43) is true. As a matter of convenience, we do a field redefinition

$$M \mapsto M - \phi I \tag{4.44}$$

such that at criticality the redefined matrix M has $\phi = 0$ on the diagonal. Next a choice of scale is made such that $y = 1$. With these choices $T(x) = x^2 - 4$. By a substitution

$$x \mapsto \xi + \xi^{-1} \tag{4.45}$$

this can be written as a complete square

$$T = (\xi - \xi^{-1})^2 \,. \tag{4.46}$$

So the entire left hand side of (4.43) is a complete square.

Note that on the r.h.s. of (4.43) the coefficients of terms proportional to x^n and higher powers of x are entirely determined by $W'(x)$. The polynomial $f_{n-1}(x)$ only serves to match the coefficients of lower powers of x. Taking the square root of (4.43) yields

$$\sqrt{T(x)G_{n-1}^2(x)} = W'(x) + \frac{1}{2}\frac{f_{n-1}(x)}{W'(x)} + \cdots = W'(x) + \mathcal{O}(x^{-1}) \tag{4.47}$$

With the substitution (4.45) , noting $x^{-1} \mapsto (\xi + \xi^{-1})^{-1} = \xi^{-1}/(1+\xi^{-2}) = \mathcal{O}(\xi^{-1})$, it is observed that (4.43) implies

$$(\xi - \xi^{-1})G_{n-1}(\xi + \xi^{-1}) = W'(\xi + \xi^{-1}) + \mathcal{O}(\xi^{-1}) \,. \tag{4.48}$$

In order that a polynomial $G_{n-1}(x)$ of degree $n-1$ exists, the right hand side of (4.48) must be divisible by $(\xi - \xi^{-1})$. Choosing the $\mathcal{O}(\xi^{-1})$ polynomial in ξ^{-1} as follows almost guarantees this:

$$(\xi - \xi^{-1})G_{n-1}(\xi + \xi^{-1}) = W'(\xi + \xi^{-1}) - 2[W'(\xi + \xi^{-1})]_- \,. \tag{4.49}$$

Here $[g(\xi)]_-$ denotes the part of the Laurent series of g with strictly negative powers of ξ. The right hand side, written as a Laurent series in ξ is of the form

$$\text{r.h.s.} = c_0 + \sum_{k=1}^{n} c_k(\xi^k - \xi^{-k}) \,. \tag{4.50}$$

This is divisible by $(\xi - \xi^{-1})$ iff $c_0 = 0$, with c_0 the coefficient of ξ^0 in the Laurent series of $W'(\xi + \xi^{-1})$. This is exactly the requirement for criticality, with the field redefinition and choice of scale that set $\phi = 0$ and $y = 1$.

In fact, a concrete expression for the coefficients of $G_{n-1}(x)$ can be found, keeping ϕ and y explicit. In order that (4.43) be true, the coefficients of each power of x must match. As noted earlier, by choosing $f_{n-1}(x)$ appropriately, n such coefficients can be matched. This leaves $n+1$ coefficients to be matched, n of which can be made to do so by an appropriate choice of $G_{n-1}(x)$.[1] In the end there remains a single relation between ϕ, y and the coefficients g_k which needs to be satisfied for (4.43) to be true.

Now let us check that the single non-trivial relation is precisely (4.38). Define

$$
\begin{aligned}
(W'(x))^2 &= \sum_{k=0}^{2n} \omega_k x^k = \sum_{k=0}^{2n} x^k \sum_{0\le r,s\le s;\, r+s=k} g_{r+1} g_{s+1} \\
G_{n-1}^2(x) &= \sum_{k=0}^{2n-2} \gamma_k x^k = \sum_{k=0}^{2n-2} x^k \sum_{0\le r,s\le n-1;\, r+s=k} \chi_{r+1}\chi_{s+1}\,.
\end{aligned}
\tag{4.51}
$$

In terms of χ_i and g_i the relevant coefficients are

$$
\begin{aligned}
\omega_{n+k} &= \sum_{l=k}^{n} g_{l+1} g_{n+k-l+1} \qquad (k=0,\dots,n) \\
\gamma_{n+k} &= \sum_{l=k+1}^{n-1} = \chi_{l+1}\chi_{n+k-l+1} \qquad (k=0,\dots,n-1)\,.
\end{aligned}
\tag{4.52}
$$

The objective is to find coefficients χ_i such that (4.43) is true, as far as the coefficients of the powers x^{n+1} to x^{2n} are concerned. That is

$$
\begin{aligned}
\omega_{2n} &= \gamma_{2n-2} \\
\omega_{2n-1} &= \gamma_{2n-3} - 2\phi\gamma_{2n-2} \\
\omega_{n+k} &= \gamma_{n+k-2} - 2\phi\gamma_{n+k-1} + (\phi^2-4y)\gamma_{n+k} \qquad (k=1,\dots,n-2)\,.
\end{aligned}
\tag{4.53}
$$

The coefficients of powers x^0 to x^{n-1} can be matched by appropriate choice of the polynomial f_{n-1} in (4.43). So in the end a single nontrivial relation remains, relating ϕ, y and the coefficients of W'.

The χ_i can be solved one by one, starting from χ_n. The top two equations in (4.53) have a solution

$$
\begin{aligned}
\chi_n &= -g_{n+1} \\
\chi_{n-1} &= -(g_n + \phi g_{n+1})\,.
\end{aligned}
\tag{4.54}
$$

The next $n-3$ equations are solved by

$$
\chi_{r+1} = -\sum_{l=0}^{n-1-r} g_{r+2+l}\phi^l\, {}_2F_1\left(-\frac{l}{2}, -\frac{l}{2}+\frac{1}{2}; 1; \frac{4y}{\phi^2}\right). \tag{4.55}
$$

[1] Note the index: this corresponds to an expansion $\sum_{i=0}^{n-1} g_{i+1}x^i$.

Subsequent χ_r's can be solved one after another because a χ_r with smaller r appears only in coefficients of lower powers of x. Thus every next condition, on the coefficients of ever lower powers of x is solved by appropriate choice of χ_r with ever lower indices. At some point, this process stops, as there is no χ_0; the coefficient of x^{-1} in $G_{n-1}(x)$ vanishes. So in order that the coefficients of x^n in (4.43) match, (4.55) must be satisfied with $\chi_0 = 0$. Note that this relation is precisely (4.38) relating ϕ to y and the coefficients of the superpotential.

4.5 Lifting solutions from U(N) to U(tN)

It is known [27] that supersymmetric vacua of a U(N) gauge theory with superpotential W can be lifted to supersymmetric vacua of a U(tN) gauge theory with exactly the same superpotential. On the level of the Seiberg-Witten polynomial, this lifting involves Chebyshev polynomials and a few other ingredients. It turns out, as we will show in this section, that it is very simple to do this at the level of Lax operators. One simply takes t copies of the Lax operator of U(N) to construct a Lax operator of U(tN) that is periodic in steps of N. This new Lax matrix is an extremum for the same superpotential, and this replica trick therefore provides a very simple picture of how to lift vacua. To show technically how this works, we start with the Lax operator for U(N)

$$M_N = \begin{pmatrix} \phi_1 & y_1 & 0 & \dots & z \\ 1 & \phi_2 & y_2 & \dots & 0 \\ 0 & 1 & \phi_3 & \dots & 0 \\ . & . & . & \dots & y_{N-1} \\ y_0 z^{-1} & . & . & 1 & \phi_N \end{pmatrix} \tag{4.56}$$

and the corresponding polynomial $P_N(x)$ in the spectral curve

$$P_N(x) = \det(xI_N - M_N) + z + \Lambda^{2N} z^{-1}\,, \tag{4.57}$$

where the energy scale $\Lambda^2 \geq 0$ sets the condition

$$\Lambda^{2N} = \prod_{i=0}^{N-1} y_i\,. \tag{4.58}$$

As explained above, for a U(tN) theory with a superpotential W of degree $d \leq N+1$ a special form of M_{tN} can be considered so that the analysis can be reduced to that of the U(N) case, with the same superpotential. This is possible when the entries of M_{tN} are periodically identified like

$$\begin{aligned} \phi_i &\equiv \phi_{i+N} \\ y_i &\equiv y_{i+N} \end{aligned} \tag{4.59}$$

so that the only non zero entries of M_{tN} are

$$\begin{aligned} (M)_{i,i} &= \phi_i \\ (M)_{i,i-1} &= 1 \end{aligned}$$

$$
\begin{aligned}
(M)_{i,i+1} &= y_i \\
(M)_{1,tN} &= z \\
(M)_{tN,1} &= y_0 z^{-1} \,.
\end{aligned}
\tag{4.60}
$$

The condition set by the energy scale is written as

$$
\prod_{i=0}^{tN-1} y_i = \Lambda^{2tN} \geq 0 \,. \tag{4.61}
$$

Note in particular that the (physical) energy scale of the U(tN) theory should be real. This condition can be satisfied by a family of t different complex valued Λ^N in the U(N) theory, $\{\Lambda^N, \eta^2\Lambda^N, \ldots, \eta^{2(t-1)}\Lambda^N\}$, with η a $2t$-th root of unity.

The polynomial $P_{tN}(x)$ in the spectral curve of M_{tN} is

$$
P_{tN}(x) = \det(xI_{tN} - M_{tN}) + z + \Lambda^{2tN} z^{-1} \,. \tag{4.62}
$$

The determinant can be calculated by considering a gauge equivalent matrix. By a gauge transformation M_{tN} can be brought into a form that is invariant under cyclic permutations of order N. We define

$$
G = \mathrm{diag}(1, z^{1/tN}, z^{2/tN}, \ldots, z^{(tN-1)/tN}) \tag{4.63}
$$

then $\tilde{M}_{tN} = G M_{tN} G^{-1}$ has factors of $z^{1/t}$ and $z^{-1/t}$ democratically distributed over the $(\tilde{M})_{i,i-1}$ and $(\tilde{M})_{i,i+1}$ entries respectively. $\tilde{M}_{tN}$ satisfies

$$
S \tilde{M}_{tN} S^{-1} = \tilde{M}_{tN} \tag{4.64}
$$

where

$$
S = \begin{pmatrix} 0 & I_{(t-1)N} \\ I_N & 0 \end{pmatrix}
$$

is the matrix that generates a cyclic permutation of order N on tN elements, $S^{-1} = S^T$.

In case the degree of the superpotential is small enough, $\deg(W) \leq N+1$, the cyclic invariance, (4.64), ensures that the equations of motion collapse to those of the U(N) theory with the same superpotential. First, because in $\tilde{M}_{tN}$ the ϕ_i and y_i appear only linearly, derivatives with respect to ϕ_{i+kN} and y_{i+kN} can be replaced with derivatives with respect to ϕ_i and y_i

$$
\begin{aligned}
\partial_{\phi_{i+N}} \mathrm{tr}\left(\tilde{M}_{tN}^{n+1}\right) &= (n+1)\mathrm{tr}\left(\tilde{M}_{tN}^{n} \partial_{\phi_{i+N}} \tilde{M}_{tN}\right) \\
&= (n+1)\mathrm{tr}\left(S\tilde{M}_{tN}^{n} S^{-1} S(\partial_{\phi_{i+N}} \tilde{M}_{tN}) S^{-1}\right) \\
&= (n+1)\mathrm{tr}\left(\tilde{M}_{tN}^{n} (\partial_{\phi_i} \tilde{M}_{tN})\right) \\
&= \partial_{\phi_i} \mathrm{tr}\left(\tilde{M}_{tN}^{n+1}\right) .
\end{aligned}
\tag{4.65}
$$

Second, in $\tilde{M}_{tN}$ the ϕ_i and y_i appear only on the diagonal and the superdiagonal respectively

$$(\tilde{M}_{tN})_{ij} = \phi_i\delta_{i,j} + y_i z^{-1/tN}\delta_{i,j-1} + z^{1/tN}\delta_{i,j+1} \tag{4.66}$$

the diagonal elements of $\tilde{M}^n_{tN}$ each depend on at most n consecutive y_i's and ϕ_i's. Therefore, all equations of motion for ϕ_1 up to ϕ_{tN} can be both mapped to equations of motion for ϕ_1 to ϕ_N and these also consistently collapse onto the equations for the first N ϕ_i's. The same holds for the equations of motion for all y_i's. Also the equation of motion for the Lagrange multiplier, (4.61) maps to (4.58).

$\tilde{M}_{tN}$ can be explicitly written as a $t\times t$ matrix of which each entry is itself one of four $N\times N$ matrices

$$\tilde{M}_{tN} = \begin{pmatrix} A & D & 0 & \dots & \dots & 0 & E \\ E & A & D & 0 & \dots & \dots & 0 \\ 0 & E & A & D & \dots & \dots & \dots \\ \dots & 0 & E & A & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & D \\ D & 0 & \dots & 0 & \dots & E & A \end{pmatrix}. \tag{4.67}$$

The $N\times N$ matrices A, D, and E are of the following form

$$A = \begin{pmatrix} \phi_1 & y_1 z^{-1/tN} & 0 & \dots & \dots & 0 \\ z^{1/tN} & \phi_2 & y_2 z^{-1/tN} & 0 & \dots & 0 \\ 0 & z^{1/tN} & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & y_{N-1}z^{-1/tN} \\ 0 & \dots & \dots & \dots & z^{1/tN} & \phi_N \end{pmatrix} \tag{4.68}$$

$$D = y_0 z^{-1/tN}\begin{pmatrix} 0_{1\times(N-1)} & 0_{(N-1)\times(N-1)} \\ 1 & 0_{(N-1)\times 1} \end{pmatrix} \tag{4.69}$$

$$E = z^{1/tN}\begin{pmatrix} 0_{1\times(N-1)} & 1 \\ 0_{(N-1)\times(N-1)} & 0_{(N-1)\times 1} \end{pmatrix}. \tag{4.70}$$

Because of (4.64), there exists a basis of simultaneous eigenvectors of $\tilde{M}_{tN}$ and S. The eigenvectors of S fall into t N-dimensional subspaces, each of which is labeled by a different t-th root of unity ω^r, $\omega^t = 1$. The eigenvalue equation for $\tilde{M}_{tN}$ is written in a basis of S eigenvectors $v_r^{(\alpha)} = (v^{(\alpha)}, \omega^r v^{(\alpha)}, \omega^{2r} v^{(\alpha)}, \dots, \omega^{(t-1)r} v^{(\alpha)})$, with $\alpha = 1, 2, \dots, N$.

For each r the eigenvalue equation $\tilde{M}_{tN} v_r^{(\alpha)} = \lambda_r^{(\alpha)} v_r^{(\alpha)}$ becomes

$$(A + \omega^r D + \omega^{-r} E) v_r^{(\alpha)} = \lambda_r^{(\alpha)} v_r^{(\alpha)} \tag{4.71}$$

and therefore

$$\det(M_{tN}) = \det(\tilde{M}_{tN}) = \prod_{r=1}^{t} \det\left(A + \omega^r D + \omega^{-r} E\right). \tag{4.72}$$

The U(tN) polynomial reads

$$
\begin{aligned}
P_{tN}(x) &= \prod_{r=1}^{t} \det\left(xI_N - A - \omega^r D - \omega^{-r} E\right) + z + \Lambda^{2tN} z^{-1} \\
&= \prod_{r=1}^{t} \left(P_n(x) - \omega^{-r} z^{1/t} - \eta^2 \Lambda^{2N} \omega^r z^{-1/t}\right) + z + \Lambda^{2tN} z^{-1} \,. \qquad (4.73)
\end{aligned}
$$

The polynomials $P(x)$ are by construction independent of z. A convenient choice to evaluate (4.73) is to take $z^{1/t} = e^{\frac{i\pi}{t}} |\Lambda^{\frac{1}{2tN}}|$. Recall that ω is a t-th root of unity such that ω^r hits all t different eigenvalues of the $\mathbb{Z}_t$ cyclic permutation matrix S. Hence $P_{tN}(x)$ can be written as

$$
\begin{aligned}
P_{tN}(x) &= \prod_{r=1}^{t} \left(P_N(x) - \eta\Lambda^N \left(e^{i\pi\frac{4r-1}{2t}} + e^{-i\pi\frac{4r-1}{2t}}\right)\right) \\
&= 2^t \eta^t \Lambda^{tN} \prod_{r=1}^{t} \left(\frac{1}{2\eta\Lambda^N} P_N(x) - \cos\left(\frac{4r-1}{2t}\pi\right)\right) . \qquad (4.74)
\end{aligned}
$$

The latter expression defines the n-th Chebyshev polynomial of the first kind, which is defined as

$$
T_t(x) = 2^{n-1} \prod_{k=1}^{t} \left(x - \cos\left(\frac{2k-1}{2t}\pi\right)\right) \qquad (4.75)
$$

with

$$
s := \max\{\sigma \in \mathbb{Z} : 2\sigma \leq t\} \qquad (4.76)
$$

equation (4.75) can also be written as

$$
\begin{aligned}
T_n(x) &= 2^{n-1} \prod_{p=1}^{s} \left(x - \cos\left(\frac{2(2p)-1}{2t}\pi\right)\right) \prod_{q=1}^{s} \left(x - \cos\left(\frac{2(2q-1)-1}{2t}\pi\right)\right) \\
&= 2^{n-1} \prod_{p=1}^{s} \left(x - \cos\left(\frac{4p-1}{2t}\pi\right)\right) \prod_{q=1}^{s} \left(x - \cos\left(\frac{4(t-q)+3}{2t}\pi\right)\right) \\
&= 2^{n-1} \prod_{r=1}^{t} \left(x - \cos\left(\frac{4r-1}{2t}\right)\right) \qquad (4.77)
\end{aligned}
$$

so

$$
P_{tN} = 2\Lambda^{tN} \eta^t T_t \left(\frac{P_N(x)}{2\eta\Lambda^N}\right) \qquad (4.78)
$$

Thus the periodic ansatz for the U(tN) theory yields t times the number of vacua found in the U(N) theory. The polynomial P_{tN} in (4.78) agrees precisely with what had been found in field theory in [27].

4.6 U(4)

We are now in a position to use the techniques of the preceding section to study our final example U(4), again with a cubic superpotential

$$W = g_1 \,\mathrm{tr}(\Phi) + \frac{g_2}{2}\,\mathrm{tr}(\Phi^2) + \frac{g_3}{3}\,\mathrm{tr}(\Phi^3)\,. \tag{4.79}$$

The Lax operator, polynomial $P(x)$, superpotential W_{eff} and equations of motion are straightforward generalizations of (4.16), (4.17), (4.18) and (4.19), with Λ^6 replaced by Λ^8. Instead of giving a lengthy and tedious analysis of the possible solutions of the equations of motion, we will simply present a solution for each of the critical points that we expect, based on the knowledge of the possible solutions in four dimensions. These are given explicitly in section 3.3 of [26].

To describe the qualitative form of the solutions, we write $W'(x) = (x - a_1)(x - a_2)$. Classically, each of the ϕ_i is equal to either a_1 or a_2. We denote the number of ϕ_i for which $\phi_i = a_j$ by N_j, so that $N_1 + N_2 = 4$. However, this does not fully specify the different solutions. For U(4) with a cubic superpotential, we know that at a minimum $P(x)^2 - 4\Lambda^8$ has two double zeroes. These double zeroes are distributed over $P(x)-2\Lambda^4$ and $P(x)+2\Lambda^4$, since these two factors cannot have a common zero. We denote by $r_\pm$ the number of double zeroes in $P(x) \pm 2\Lambda^4$, so that $r_+ + r_- = 2$.

When all ϕ_i are equal, with $(N_1, N_2) = (4, 0)$ or $(0, 4)$, we have a maximally confining vacuum, and these were already described in detail in (4.4).

When $(N_1, N_2) = (2, 2)$, we can use the results in (4.5) to find solutions by lifting solutions in U(2) to U(4). One easily finds that these solutions have $(r_+, r_-) = (2, 0)$ or $(0, 2)$. In addition, they necessarily have $(\phi_1, \phi_2, \phi_3, \phi_4) = (a_1, a_2, a_1, a_2)$ or (a_2, a_1, a_2, a_1).

However, this does not exhaust all solutions with $(N_1, N_2) = (2, 2)$. There are also solutions that have $(r_+, r_-) = (1, 1)$. In addition, we have not yet considered solutions with $(N_1, N_2) = (1, 3)$ or $(3, 1)$. It turns out that all missing solutions are part of one family, that can be described as follows. First, by an overall rescaling and by shifting x by a constant we can always choose $W(x)$ so that

$$W'(x) = x^2 + x\frac{\Lambda^4}{a^3} - a^2\,, \tag{4.80}$$

for some parameter a. The solution is then

$$\begin{aligned}
\phi_1 &= \frac{a^2}{\phi_3} & \qquad y_0 &= a^2\frac{\phi_3+\phi_4}{\phi_3} \\
\phi_2 &= \frac{a^2}{\phi_4} & \qquad y_1 &= a^2 + \frac{a^4}{\phi_3\phi_4} \\
\phi_3 &= \phi_3 & \qquad y_2 &= a^2\frac{\phi_3+\phi_4}{\phi_4} \\
\phi_4 &= \phi_4 & \qquad y_3 &= a^2 + \phi_3\phi_4 \\
L &= \frac{\Lambda^4}{a} & &
\end{aligned} \tag{4.81}$$

with ϕ_3, ϕ_4 subject to the constraint

$$(\phi_3 + \phi_4)(a^5 + a^3\phi_3\phi_4) + \phi_3\phi_4\Lambda^4 = 0\,. \tag{4.82}$$

As explained in [26], there are two classical limits, one is $\Lambda \to 0$ while keeping a fixed, the other is $\Lambda \to 0$ while keeping Λ^4/a^3 fixed. The first corresponds to $(N_1, N_2) = (2,2)$ and $(r_+, r_-) = (1,1)$, the second to $(N_1, N_2) = (1,3)$ or $(3,1)$.

We have accounted for all vacua that we expect in four dimensions. One interesting feature of the solutions is that in the case with $(N_1, N_2) = (2,2)$, all classical limits have $\phi_1 = \phi_3$ and $\phi_2 = \phi_4$, but there is no solution whose classical limit obeys $\phi_1 = \phi_2$ and $\phi_3 = \phi_4$. It therefore appears that one should be careful in choosing the right ordering of the eigenvalues, not all orderings will give rise to a solution of the quantum equations of motion. This is not a contradiction, since the choice of Lax matrix breaks the S_N symmetry to Z_N, and there is no symmetry that arbitrarily permutes the eigenvalues.

The solution in (4.81) has one free parameter, which corresponds to the extra U(1) that appears when breaking $\mathrm{U}(4) \to \mathrm{U}(2) \times \mathrm{U}(2)$.

5. The semi-classical expansion

In the previous section we presented several examples of superpotentials and their extrema. The solutions depend in a non-trivial way on Λ, and it is important to understand the nature of the semi-classical expansion, certainly if we want to compare our results to direct field theory calculations in three dimensions. A precise understanding of the semi-classical expansion is probably also important in order to understand the relation to the 4d description in terms of gluino bilinear superfields, as we will discuss in section 7.

What is the general structure of the semiclassical expansion that one would expect to find? In four dimensions in a situation where the the gauge group is classically broken to $\prod_i \mathrm{U}(N_i)$ the effective superpotential explicitly takes the form of a semiclassical expansion. In addition to the Veneziano-Yankielowicz superpotentials (2.13) for each unbroken $\mathrm{U}(N_i)$, there are many additional terms coming from the planar diagrams of the matrix model. These are polynomial and give ultimately rise to an expansion in positive but possibly fractional powers of Λ. The low-energy scales Λ_i of the unbroken $\mathrm{U}(N_i)$ that appear in the Veneziano-Yankielowicz superpotential (2.13) are obtained by scale matching and given by

$$\Lambda_i^{3N_i} = g_n^{N_i}\Lambda^{2N}\prod_{j\neq i}(a_j - a_i)^{N_i - 2N_j}\,, \tag{5.1}$$

with a_i and g_n defined in (2.6).

On $R^3 \times S^1$, the Veneziano-Yankielowicz superpotential is no longer appropriate, the relevant superpotential is instead given by (2.17) or equivalently (2.21). Therefore, we expect that the superpotential on $R^3 \times S^1$ should admit an expansion of the form

$$W = \sum_i \sum_{j=0}^{N_i - 1} y_j^{(i)} + \text{higher order terms} \tag{5.2}$$

where the $y_j^{(i)}$ are analogues of the variables y_i for each of the classically unbroken $\mathrm{U}(N_i)$ groups, and they should therefore obey

$$\prod_{j=0}^{N_i-1} y_j^{(i)} = \Lambda_i^{3N_i} \tag{5.3}$$

for each i. The superpotential in the form (5.2) only depends on variables y_i, not on the eigenvalues ϕ_i of Φ. In addition, it depends on a choice of classical vacuum, whereas the superpotential (3.4) with the ϕ_i included describes all vacua. Therefore, a description like (5.1) can only emerge after we integrate out the ϕ_i.

To illustrate how an expansion like (5.2) can emerge, we consider the case $\mathrm{U}(4) \to \mathrm{U}(2) \times \mathrm{U}(2)$, with superpotential $W(x) = x^3/3 - a^2 x$, so that $W'(x) = (x-a)(x+a)$. As a first step, we integrate out the ϕ's in the quantum superpotential. This can be done explicitly, the solutions for the ϕ_i read

$$\begin{aligned} \phi_1 &= \sqrt{a^2 - y_0 - y_1} & \qquad \phi_3 &= \sqrt{a^2 - y_2 - y_3} \\ \phi_2 &= -\sqrt{a^2 - y_1 - y_2} & \qquad \phi_4 &= -\sqrt{a^2 - y_0 - y_3}\,. \end{aligned}$$

Notice that we chose the signs of the square roots in such a way that in the classical limit $y_i \to 0$ we indeed end up in a vacuum where $\mathrm{U}(4)$ is broken to $\mathrm{U}(2) \times \mathrm{U}(2)$. Therefore integrating out the ϕ_i also involves the selection of a classical vacuum configuration. If we insert the values for ϕ_i in the superpotential, and expand the result to second order in y, we get

$$W_{\mathrm{eff}} = L \log\left(\frac{\Lambda_4^8}{y_0 y_1 y_2 y_3}\right) - \frac{1}{2a}(y_0 - y_2)(y_1 - y_3) + \cdots .$$

If we next integrate out L, this becomes

$$W_{\mathrm{eff}} = -\frac{1}{2a}\left(y_0 y_1 + \frac{\Lambda^8}{y_0 y_1} - y_1 y_2 - \frac{\Lambda^8}{y_1 y_2}\right). \tag{5.4}$$

Interestingly, this depends on only two independent variables, namely $y_0 y_1$ and $y_1 y_2$, and therefore it is already of the form (5.2). Indeed, if we define

$$y_0^{(1)} = -\frac{y_0 y_1}{2a}\,, \qquad y_1^{(1)} = -\frac{\Lambda^8}{2a y_0 y_1}\,, \qquad y_0^{(2)} = \frac{y_1 y_2}{2a}\,, \qquad y_1^{(2)} = \frac{\Lambda^8}{2a y_1 y_2} \tag{5.5}$$

then (5.4) is of the form (5.2), and $y_0^{(1)} y_1^{(1)} = \Lambda_1^6 = \Lambda^8/(2a)^2$, $y_0^{(2)} y_1^{(2)} = \Lambda_2^6 = \Lambda^8/(2a)^2$, completely in agreement with (5.3) and (5.1).

Of course, the above result is simply the semiclassical result at leading order. It would be very interesting to go beyond the leading order, and to understand in detail how the expansion is organized. We have not studied this in detail, but expect the following. In general the value of the superpotential is invariant under the flows (3.3) of the integrable system. Some of the flows are stationary at the extremum, but some are not, and that is why there is a $k-1$ complex parameter family of minima (see also the discussion in section 3.1.) If we could redefine our y variables in such a way that $N-k$ of them are independent of the $k-1$ flows that do not degenerate, then the superpotential should be a non-trivial function of these $N-k$ variables only. This is exactly the number of independent variables that appears in the semiclassical expansion (5.2). Hence we expect that the semiclassical expansion appears naturally by integrating out the variables ϕ_i and L, and by subsequently redefining the complex variables y_i in a suitable way, exactly as in our example above.

To conclude this section, we illustrate in the case of U(4) with a cubic potential $W(x) = x^3/3 - a^2 x$ how the solutions found in section 4.6 can be expanded in (possibly fractional) powers of Λ.

For the maximally confining case, with $(N_1, N_2) = (4, 0)$, we have

$$\begin{aligned} \phi_i &= a - \omega\frac{\Lambda^2}{a} + \cdots \\ y_i &= \omega\Lambda^2 + \cdots \\ L &= 2a\omega\Lambda^2 + \cdots \end{aligned} \tag{5.6}$$

with ω a fourth root of unity.

For $(N_1, N_2) = (2, 2)$ with $(r_+, r_-) = (2, 0)$ we find

$$\begin{aligned} \phi_1 = \phi_3 = -\phi_2 = -\phi_4 &= a - \left(\xi + \frac{1}{\xi}\right)\frac{\Lambda^2}{2a} + \cdots \\ y_0 = y_2 &= \xi\Lambda^2 + \cdots \\ y_1 = y_3 &= \frac{1}{\xi}\Lambda^2 + \cdots \\ L &= 0\,. \end{aligned} \tag{5.7}$$

Here, and in the solutions below, ξ indicates the free parameter that is related to the additional unbroken U(1) that one gets in the corresponding vacuum solution.

For $(N_1, N_2) = (2, 2)$ with $(r_+, r_-) = (1, 1)$ the expansion reads

$$\begin{aligned} \phi_1 &= a - \left(\xi - \frac{1}{\xi}\right)\frac{\Lambda^2}{2a} + \cdots & y_0 &= \xi\Lambda^2 + \cdots \\ \phi_2 &= -a - \left(\xi + \frac{1}{\xi}\right)\frac{\Lambda^2}{2a} + \cdots & y_1 &= -\tfrac{1}{\xi}\Lambda^2 + \cdots \\ \phi_3 &= a + \left(\xi - \frac{1}{\xi}\right)\frac{\Lambda^2}{2a} + \cdots & y_2 &= -\xi\Lambda^2 + \cdots \\ \phi_4 &= -a + \left(\xi + \frac{1}{\xi}\right)\frac{\Lambda^2}{2a} + \cdots & y_3 &= \tfrac{1}{\xi}\Lambda^2 + \cdots \\ L &= \frac{\Lambda^4}{a} + \cdots\,. \end{aligned} \tag{5.8}$$

This result differs considerably from (5.7), illustrating the fact that these are two different solutions.

Finally, for $(N_1, N_2) = (3, 1)$ the result is an expansion in

$$\epsilon = \left(\frac{\Lambda^4}{2a}\right)^{1/3}, \tag{5.9}$$

which reads

$$\begin{aligned}
\phi_1 &= a - \frac{\epsilon}{\xi} + \cdots & y_0 &= \frac{2a\epsilon}{\xi} + \cdots \\
\phi_2 &= a - \frac{\epsilon^2}{a} + \cdots & y_1 &= \epsilon^2 + \cdots \\
\phi_3 &= a - \epsilon\xi + \cdots & y_2 &= \epsilon^2 + \cdots \\
\phi_4 &= -a + \epsilon(\xi + \tfrac{1}{\xi}) + \cdots & y_3 &= 2a\epsilon\xi + \cdots \\
L &= 2a\epsilon^2 + \cdots\,.
\end{aligned} \tag{5.10}$$

6. Integrable systems interpretation

The Lax matrix of the periodic Toda chain played an essential role in the construction of the effective superpotential in three dimensions. One of the motivations of this work was to try to find a direct relation between the periodic Toda chain and the Dijkgraaf-Vafa matrix model. We have not yet succeeded in finding a direct relation, but nevertheless we have found that the supersymmetric vacua in three dimensions have a very nice interpretation in terms of the Toda integrable system. To explain this, we first need to define the operator $\mathcal{L}$ that appears in (3.3),which acts on powers of the Lax matrix M; M, given in (3.5), depends on a spectral parameter z. Therefore, any power of M can be written as a series in z as

$$M^k = \sum_r z^r M^k_{(r)}\,. \tag{6.1}$$

Next, we define M^k_+ as the sum of the upper diagonal part of $M^k_{(0)}$ plus $\sum_{r<0} z^r M^k_{(r)}$, M^k_- as the sum of the lower diagonal part of $M^k_{(0)}$ plus $\sum_{r>0} z^r M^k_{(r)}$, and finally M^k_0 as the diagonal part of $M^k_{(0)}$. Then

$$\mathcal{L}(M^k) \equiv M^k_- + M^k_0 - M^k_+\,. \tag{6.2}$$

One can interpret powers of M also as elements of the affine Lie algebra $\widehat{gl}_N$, with z^{-1} corresponding to the extra affine root of the extended Dynkin diagram. Then (6.2) is nothing but the statement that all positive affine roots change sign. Alternatively, one can embed $\widehat{gl}_N$ in gl_∞, by extending every matrix to a periodic infinite matrix with period N (i.e. $A_{i+N,j+N} = A_{ij}$), and by replacing z by the shift matrix D^N with $D_{ij} = \delta_{i-1,j}$. Then (6.2) amounts to just changing the sign of the upper triangular part of the corresponding infinite matrix.

The equations of motion derived from (3.4) imply that the flow generated by $W(M)$ degenerates. This translates to the equations

$$[M, W'(M)_+ - W'(M)_-] = 0\,, \qquad W'(M)_0 = 0\,. \tag{6.3}$$

A second observation, which follows from (3.7), is that

$$[M, P_N(M)_+ - P_N(M)_-] = 0\,. \tag{6.4}$$

The two equations (6.3), (6.4) define a reduction of the integrable Toda system. As we will show in [23], these equations imply the existence of polynomials H_{N-k}, T_{2k}, G_{n-k}, f_{n-1}, U_k of degrees $N-k$, $2k$, $n-k$, $n-1$, k for $0 < k \leq n$ such that

$$\begin{aligned}(W'(M)_+ - W'(M)_-)^2 &= W'(M)^2 + f_{n-1}(M) &(6.5)\\ W'(M)_+ - W'(M)_- &= G_{n-k}(M)(U_k(M)_+ - U_k(M)_-) &(6.6)\\ (P_N(M)_+ - P_N(M)_-)^2 &= P_N(M)^2 - 4\Lambda^{2N} &(6.7)\\ P_N(M)_+ - P_N(M)_- &= H_{N-k}(M)((U_k(M)_+ - U_k(M)_-) &(6.8)\\ (U_k(M)_+ - U_k(M)_-)^2 &= T_{2k}(M)\,. &(6.9)\end{aligned}$$

This system of equations demonstrates not only that the extrema of the superpotential (3.4) are in exact one-to-one correspondence with the four-dimensional results, it also shows that we can construct the four-dimensional equations (2.9) and (2.10) directly in terms of the Lax matrix. In particular, the matrix model resolvent [3] $2R(z) = -\sqrt{W'(z)^2 + f_{n-1}(z)} + W'(z)$ satisfies the very simple equation

$$R(M) = W'(M)_-\,. \tag{6.10}$$

Actually, equation (6.3) is tantalizingly close to a similar equation that can be derived for matrix integrals of the form

$$\int d\Phi e^{-W(\Phi)}\,. \tag{6.11}$$

If one defines a set of orthogonal polynomials with respect to the measure $e^{-W(x)}$, of the form $p_n = x^n + \cdots$, one can construct a semi-infinite matrix Q that acts on p_n as multiplication by x. This semi-infinite matrix is tridiagonal (i.e. $Q_{ij} = 0$ for $|i-j| > 1$), and obeys the "string equation" (see for example [40] and references therein)

$$[Q, -\frac{1}{2}(W'(Q)_+ - W'(Q)_-)] = 1\,. \tag{6.12}$$

The only difference between (6.3) and (6.12) is that one is replaced by zero. Equation (6.12) is also the equation that gives rise to the string equation in the matrix model description of minimal models coupled to gravity. Though (6.3) and (6.12) are very similar, we have not found a direct map between Q and M. Whereas Q is relevant for the orthogonal polynomials defined with respect to $e^{-W(x)}$, M seems to define orthogonal polynomials for a measure which coincides with the gauge theory resolvent instead. Various other relations between matrix integrals and the Toda lattice equations are discussed in e.g. [41, 42].

The definition of the gauge theory resolvent (the generating functional of $\langle \mathrm{Tr}(\Phi^m)\rangle$) requires some discussion. For sufficiently large m, we can no longer have the identity $\langle \Phi^m \rangle = \mathrm{Tr}(M^m)$, because the right hand side will start to depend non-trivially on the spectral parameter. The resolution is to use the fact that given a Lax matrix M_N of size $N \times N$, there is a corresponding Lax matrix M_{tN} of size $tN \times tN$ which is constructed using the replica procedure given in section 4.5. For small m, $\mathrm{Tr}(M_N^m) = t^{-1}\,\mathrm{Tr}(M_{tN}^m)$, but for larger values of m the right hand side starts to depend on the spectral parameter, whereas the left hand side does not. We therefore propose

$$\langle \Phi^m \rangle = \lim_{t\to\infty} \frac{1}{t} \operatorname{Tr}(M_{tN}^m) \tag{6.13}$$

as the right definition for arbitrary m. Note that the replica procedure yields

$$P_{tN}(x) = 2\Lambda^{tN} T_t \left(\frac{P_N(x)}{2\Lambda^N} \right). \tag{6.14}$$

With this expression for $P_{tN}(x)$ and the identities for Chebyshev polynomials

$$\begin{aligned} T_n'(x) &= nU_{n-1}(x) \\ T_n^2(x) - 1 &= (x^2-1)U_{n-1}^2(x) \end{aligned} \tag{6.15}$$

it is easy to see that this proposition yields a resolvent that agrees with the field theory result of [5],

$$\langle \operatorname{tr} \frac{1}{x-\Phi} \rangle = \frac{P'(x)}{\sqrt{P^2(x) - 4\Lambda^{2N}}}. \tag{6.16}$$

Therefore, M_{tN} for large t provides a master field for Φ. For the simple case of a quadratic superpotential (see section 4.3) M takes a simple form, and M_{tN} for infinite t can be written in a simple way in terms of Cuntz variables. In this way it also provides a master field for the gaussian matrix model, as observed in [43], but this appears to be a coincidence that happens only for quadratic superpotentials. Whether there is a direct way other than (6.10) to extract the matrix model resolvent remains to be seen.

A further discussion and a proof of relations (6.5)-(6.9) will be given in [23].

7. Interpretation of glueball fields

Though we have seen that the matrix model resolvent has a direct interpretation in terms of the Lax matrix, this does not yet explain how to relate it to the four-dimensional superpotential (2.12). One thing that is easy to do is to figure out what the interpretation of the chiral superfield $S = \sum_i S_i$ is. According to [44], the S-dependence can be derived from the value of the superpotential at the minimum by integrating it back in. The integrating in procedure amounts to replacing $W_{\min}(\Lambda)$ by $W_{\min}(\Omega) + S\log(\Lambda^{2N}/\Omega^{2N})$. The value $2N$ in the exponent comes from the coefficient of the β-function in 3d. Integrating out S reproduces $W_{\min}(\Lambda)$, and the S-dependence is found by integrating out Ω.

Now the only Λ-dependence of the effective action is through the Lagrange multiplier term (3.6), and if we integrate out Ω from

$$L \log \left(\frac{\Omega^{2N}}{\prod_{i=0}^{N-1} y_i} \right) + S \log \left(\frac{\Lambda^{2N}}{\Omega^{2N}} \right) \tag{7.1}$$

we obtain

$$S \log \left(\frac{\Lambda^{2N}}{\prod_{i=0}^{N-1} y_i} \right) \tag{7.2}$$

which is the same as the original Lagrange multiplier term (3.6), except that L has been replaced by S. This shows that in general the Lagrange multiplier L can be identified with $S = \sum_i S_i$. To illustrate this fact, we will next demonstrate how to recover the semiclassical expansion in four dimensions in terms of S from the three-dimensional superpotential (3.4). We will consider two cases: U(2) in the maximally confining case and U(4) breaking to U(2) $\times$ U(2). After that, we will return to the general case.

7.1 U(2)

For convenience we specialize to the superpotential

$$W(x) = \frac{x^3}{3} - a^2 x \,, \tag{7.3}$$

which has extrema at $x = \pm a$. Since we are going to study the maximally confining case there is only one chiral superfield S, which therefore has to be equal to the Lagrange multiplier L:

$$L = S \,.$$

As explained above, the S-dependence can be recovered from the superpotential

$$W_{\text{eff}} = W(\phi_1) + W(\phi_2) + (y_0 + y_1)(\phi_1 + \phi_2) + S \log\left(\frac{\Lambda^4}{y_0 y_1}\right)$$

by integrating out the ϕ's and the y's degrees of freedom. This means we have to solve for the extrema of the superpotential in terms of the ϕ's and y's, so we can simply use the results of section 4.1.[2] The only thing we have to do is to choose the vacuum. In the classical limit ϕ is $\pm a$, we pick $\phi_{\text{clas}} = a$. To integrate out the ϕ's we pick the corresponding solution from section 4.1 (with $g_3 = 1, g_2 = -a^2, g_1 = 0$)

$$\begin{aligned} y &= y_0 = y_1 \\ \phi &= \phi_1 = \phi_2 = \sqrt{a^2 - 2y} \,. \end{aligned} \tag{7.4}$$

Plugging this into the superpotential leaves us with

$$W_{\text{eff}} = -\frac{4}{3}(a^2 - 2y)^{\frac{3}{2}} + S \log\left(\frac{\Lambda^4}{y^2}\right) . \tag{7.5}$$

The next step is to integrate out the y's

$$\frac{\partial W}{\partial y} = 0 \Rightarrow S^2 = 4y^2(a^2 - 2y) \,. \tag{7.6}$$

To solve (7.6) we write

$$\xi = \frac{S}{4a^3} \qquad \text{and} \qquad y(S) = A(\xi)\frac{S}{2a} \,.$$

[2] Since we do not want to integrate out S we have to be careful not to use the equation of motion for S: $y_0 y_1 = \Lambda^4$.

Then (7.6) can be written as

$$A^2(\xi) - 4\xi A^3(\xi) = 1\,.$$

In principle there are three solutions for $A(\xi)$, however not all solutions have the right classical limit ($\Lambda \to 0$). From the solution presented in section 4.1 we learn that in the classical limit we need to have $y = S/2a$, therefore:

$$y = \frac{S}{2a} + \mathcal{O}(S^2)\,, \qquad \text{hence} \qquad A(0) = 1\,.$$

The solution for $A(\xi)$ (with $A(0) = 1$) is (see for example [45])

$$A(\xi) = \sum_{n=0}^{\infty} \frac{2^{2n}}{n+1} \binom{\frac{3n-1}{2}}{n} \xi^n = \frac{1}{12\xi} + \frac{1}{6\xi} \sin\left(\arcsin\left(\frac{216\xi^2 - 1}{3}\right)\right)$$

yielding the following expression for y

$$\begin{aligned} y(S) &= \frac{S}{2a} \sum_{n=0}^{\infty} \frac{2^{2n}}{n+1} \binom{\frac{3n-1}{2}}{n} \left(\frac{S}{4a^3}\right)^n = -\frac{a^2}{6}\left(-1 + 2\sin\left(\frac{1}{3}\arcsin\left(1 - \frac{27S^2}{2a^6}\right)\right)\right) \\ &= \frac{S}{2a} + \frac{S^2}{4a^4} + \frac{5S^3}{16a^7} + \mathcal{O}(S^4)\,. \end{aligned}$$

Substituting this solution into the effective superpotential gives us, in principle, a closed expression valid to all orders in S. However the form of this expression is not particularly illuminating and therefore we expand the superpotential in S

$$W_{\text{eff}} = -\frac{4a^3}{3} + 2S\left(1 - \log\left(\frac{S}{\Lambda^2 m}\right)\right) - \left(\frac{S^2}{2a^3} + \frac{S^3}{3a^6} + \frac{35S^4}{96a^9} + \cdots\right), \qquad (7.7)$$

here $m = 2a$ is the mass of the fluctuations of ϕ around the classical extremum $\phi = a$.

Let us compare this with the four dimensional answer [46], in that case we would write (for gauge group $\mathrm{U}(N)$)

$$W_{\text{eff}} = NS\left(1 - \log\left(\frac{S}{\Lambda^3}\right)\right) - NS\log\left(\frac{\Lambda}{m}\right) - N\frac{\partial \mathcal{F}}{\partial S}\,.$$

For a superpotential $W = \frac{g}{3}\Phi^3 + \frac{m}{2}\Phi^2$ (i.e. $m = 2a, g = 1$) the function $\mathcal{F}$ is given by

$$\mathcal{F} = \frac{2}{3}\frac{g^2}{m^3}S^3 + \frac{8}{3}\frac{g^3}{m^6}S^4 + \frac{56}{3}\frac{g^4}{m^9}S^5 + \cdots \Rightarrow 2\frac{\partial \mathcal{F}}{\partial S} = \frac{S^2}{2a^3} + \frac{S^3}{3a^6} + \frac{35S^4}{96a^9} + \cdots\,.$$

So we see that equation (7.7) is in good agreement with the four-dimensional answer.

7.2 $\mathrm{U}(4) \to \mathrm{U}(2) \times \mathrm{U}(2)$

In this section we consider the gauge group $\mathrm{U}(4)$ and the superpotential as in equation (7.3). We expand around the following vacuum

$$\phi_{1,3} = a \qquad \phi_{2,4} = -a\,.$$

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Classically this vacuum breaks the U(4) to a U(2) $\times$ U(2) symmetry. So in this case there are two chiral superfields involved: S_1 and S_2. However, we can only integrate in the sum of these, since $L = S = S_1 + S_2$. To integrate in S we use the same approach as in the previous section, we integrate out the ϕ's and y's.

The first steps are parallel to the calculation done in section 5. We integrate out the ϕ's first, allowing us to express the ϕ's in terms of the y's

$$\begin{aligned} \phi_1 &= \sqrt{a^2 - y_0 - y_1} \qquad & \phi_3 &= \sqrt{a^2 - y_2 - y_3} \\ \phi_2 &= -\sqrt{a^2 - y_1 - y_2} \qquad & \phi_4 &= -\sqrt{a^2 - y_0 - y_3} \,. \end{aligned}$$

We plug this into the superpotential which is then expressed in terms of the y's only. In principle we can proceed to integrate out the y's, however the algebra involved is rather messy, therefore we choose to expand the superpotential as a power series in the y's. Up to second order the superpotential then reads

$$W_{\text{eff}} = S \log\left(\frac{\Lambda_4^8}{y_0 y_1 y_2 y_3}\right) - \frac{1}{m}(y_0 - y_2)(y_1 - y_3) + \cdots \,.$$

Integrating out the y's yields

$$y_0 = \frac{Sm}{2y_3}\,, \qquad y_1 = -y_3\,, \qquad y_2 = -\frac{Sm}{2y_3}$$

and leads to the following effective superpotential

$$W_{\text{eff}} = S\left(2 + \log\left(\frac{4\Lambda_4^8}{m^2 S^2}\right)\right).$$

The scale Λ_4, corresponding to the U(4), can be related to the scales of the U(2)'s Λ_2 (we will write $\Lambda_2^3 = m\Lambda^2$)

$$\Lambda_2^6 = \frac{\Lambda_4^8}{m^2} = m^2\Lambda^4 \Rightarrow \Lambda_4^2 = m\Lambda\,.$$

So we finally have

$$W_{\text{eff}} = 2S\left(1 - \log\left(\frac{S}{2m\Lambda^2}\right)\right). \tag{7.8}$$

In order to compare this with the four dimensional result [46]

$$W_{\text{eff}} = 2\left(-S_1 \log\left(\frac{S_1}{m\Lambda^2}\right) - S_2 \log\left(\frac{S_2}{m\Lambda^2}\right) + S_1 + S_2 + \cdots\right)$$

we should express this effective superpotential in terms of $S = S_1 + S_2$, so we have to integrate out $S_1 - S_2$. Since we have expanded the action only to first order in the chiral superfields, the two chiral superfields don't mix and integrating out $S_1 - S_2$ is trivial. The result is that $S_1 = S_2 = S/2$, substituting this in the superpotential gives us back equation (7.8).

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7.3 Going from R^4 to $R^3 \times S^1$

Once we identify the Lagrange multiplier L with the glueball field $S = \sum_i S_i$, there is a concrete procedure to find the superpotential as a function of S from (3.4). One may wonder whether one can also go back and start with the result on R^4 and construct the superpotential on $R^3 \times S^1$. We don't know whether this can be done in general, but a step in this direction is to show how one can obtain (2.17) from (2.13). The procedure is very similar to the path integral derivation of 2d mirror symmetry given in [47]. Starting with the Veneziano-Yankielowicz superpotential

$$W_{VY}(S) = S\left[\log\left(\frac{\Lambda^{3N}}{S^N}\right) + N\right] \tag{7.9}$$

we first write it in a form as if it were built out of U(1) pieces rather than U(N),

$$W_{VY} \sim \sum_{t=1}^{N} S_t\left[\log\left(\frac{\Lambda^3}{S_t}\right) + 1\right] + \sum_{t=1}^{N-1} Z_t(S_t - S_{t+1})\,. \tag{7.10}$$

If we integrate out the variables Z_t, all S_t are identified, and we are back at the form (7.9) of the superpotential. However, we proceed by integrating out the S_t from (7.10) instead. This yields

$$W \to \Lambda^3(e^{Z_1} + e^{Z_2 - Z_1} + \cdots + e^{-Z_{N-1}}) \tag{7.11}$$

which is indeed precisely of the form (7.10). It would be very interesting to understand whether and how these transformations can be generalized, perhaps in the spirit of 2d mirror symmetry, to more complicated situations.

7.4 Interpretation of the individual S_i

A full reconstruction of the superpotential (2.12) from (3.4) requires us to not only find the right interpretation of $S = \sum S_i$, but also of the individual S_i. To do this the nature of the semiclassical expansion discussed in section 5 is probably crucial. If we could write the superpotential in the form (5.2) with constraints (5.3), we could try to impose the constraints (5.3) using various Lagrange multiplier fields L_i similar as in (3.6), and it would then be natural to identify those with the S_i. To lowest order, this would simply boil down to integrating in the S_i in each individual gauge group and therefore correctly reproduce the Veneziano-Yankielowicz superpotentials (2.13). We leave a further study of this to future work.

8. Conclusions

In this paper we have described the low-energy effective superpotential for supersymmetric gauge theories on $R^3 \times S^1$. We have shown that it yields precisely the same vacuum structure as one obtains on R^4, but the relation between the two is highly non-trivial. The results are a first step towards a direct derivation of the integrable system that underlies the Dijkgraaf-Vafa matrix model from field theory.

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We believe that the formulation of the theory on $R^3 \times S^1$ offers some advantages over the formulation on R^4. For instance, the generalization of the periodic Toda chain to arbitrary gauge groups is known [16], and therefore the present formalism should also be applicable to groups like G_2 and E_6, for which the Dijkgraaf-Vafa matrix model has not yet been worked out. We also know the relevant integrable system for various other gauge theories,as summarized in [17], such as $N = 2$ theories with matter [48], $N = 4$ super Yang-Mills theory [49], certain conformally invariant $N = 2$ theories with gauge groups of quiver type [49, 21], and even for some 5d theories [50, 51]. Ultimately, we would like to understand in all these cases the nature of the reduction of the integrable system in a supersymmetric vacuum, thereby generalizing the results in section 6.

In string theory, there is no obvious way to describe 3d field theory results using topological string theory, since this would require some 7d topological string theory. The integrable system can however in some cases be extracted from string theory using dualities and suitable brane configurations [52], and it is worthwhile to explore this connection further.

The integrable system itself plays a crucial role in this discussion. There are many physical properties that beg for a nice explanation in terms of the integrable system, such as for example the loop equations and generalized Konishi anomaly of [5]. At the same time, there are many features of the integrable system we have not yet used, such as the description of its solutions in terms of algebraic-geometric data [53], and such as the existence of additional flows associated to Whitham times (see e.g. [54] and references therein). The latter may help in finding the correct interpretation of the gluino bilinear superfields S_i.

We found it particularly elegant that lifting vacua from U(N) to U(tN) had such a nice interpretation in terms of a simple replica trick for the Lax matrix. In this context it is amusing to observe that there is a close relation between replica tricks, random matrix theory and the Toda lattice hierarchy [55], but whether that is of any relevance to the present discussion remains to be seen.

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A. A recurrence relation for $P_N(x)$

In this appendix we derive a recurrence relation for characteristic polynomials of the following type

$$P_N(x) = \det(x1_N - \Phi) = \begin{vmatrix} x-\phi_1 & -y_1 & 0 & . & . & . & 0 & -z \\ -1 & x-\phi_2 & -y_2 & 0 & . & . & . & 0 \\ 0 & -1 & x-\phi_3 & -y_3 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & -1 & x-\phi_{N-2} & -y_{N-1} \\ -\frac{y_0}{z} & 0 & . & . & . & . & -1 & x-\phi_{N-1} \end{vmatrix}. \tag{A.1}$$

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This determinant can be expressed in terms of determinants of the following form

$$G_N(x) = \begin{vmatrix} x-\phi_1 & -y_1 & 0 & . & . & . & 0 & 0 \\ -1 & x-\phi_2 & -y_2 & 0 & . & . & . & 0 \\ 0 & -1 & x-\phi_3 & -y_3 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & -1 & x-\phi_{N-2} & -y_{N-1} \\ 0 & 0 & . & . & . & . & -1 & x-\phi_{N-1} \end{vmatrix} . \tag{A.2}$$

Expanding $P_N(x)$ along the bottom line and keeping only z-independent terms we get

$$\begin{aligned} P_N(x) &= (x-\phi_{N-1})G_{N-1}(x) - (-1)(-y_{N-1})G_{N-2}(x) + \\ &\quad +(-1)^{N+1}(-\frac{y_0}{z})(-1)^N(-z)G^+_{N-2}(x) \\ &= (x-\phi_{N-1})G_{N-1}(x) - y_{N-1}G_{N-2}(x) - y_0 G^+_{N-2}(x)\,, \end{aligned} \tag{A.3}$$

here $G^+_N(x)$ is equal to $G_N(x)$ with shifted ϕ's and y's (i.e. $\phi_i \to \phi_{i+1}, y_i \to y_{i+1}$).

The $G_N(x)$ and $G^+_N(x)$ are tri-diagonal and therefore satisfy the recurrence relations

$$G_N(x) = (x-\phi_{N-1})G_{N-1}(x) - y_{N-1}G_{N-2}(x) \tag{A.4}$$

$$G^+_N(x) = (x-\phi_N)G^+_{N-1}(x) - y_N G^+_{N-2}(x)\,. \tag{A.5}$$

As a special case, take all the y's and ϕ's equal, then $G^+_N(x) = G_N(x)$ and the recurrence relation for $P_N(x)$ is

$$P_N(x) = (x-\phi)G_{N-1} - 2yG_{N-2} = G_N(x) - yG_{N-2}\,,$$

in this case it is easy to show that the P_N satisfies the same recurrence relation as the G_N

$$P_N(x) = (x-\phi)P_{N-1}(x) - yP_{N-2}(x)\,, \tag{A.6}$$

which is, up to some rescaling, the Chebyshev recurrence relation. Therefore the $P_N(x)$ (with all y's and ϕ's equal) are given by the Chebyshev polynomials of the first or of the second kind.

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